MULTI-DIMENSIONAL PARTIAL DIFFERENCE OPERATORS IN HEAT FLOW

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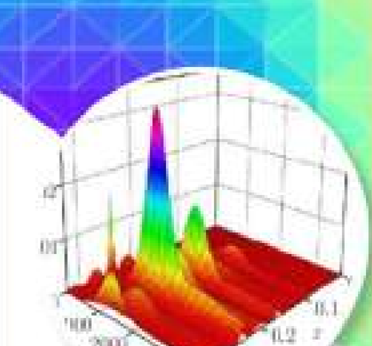

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April 30, 2024

Published by
Royal Book Publishing
21/11, K M Nagar,
Ayodhiyapatinam, Salem.
Tamil Nadu-636103
Cell:+91 7598141199
Email: contact@royalbookpublishing.com
Website: www.royalbookpublishing.com

Published in India.

International Standard Book Number (ISBN) : 9788119821372

Price: Rs. 373/-

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Chapter 1

Introduction

1.1 Difference operators and equations

A difference equation is an equation that contains sequence differences. There are various types of difference equations namely ordinary, delay, advanced, neutral, quasilinear, half linear, etc. These equations occur in numerous settings and forms, both in mathematics itself and its applications to Biology, Computer Science, Digital Signal Processing, Economics, Statistics and other fields.

The theory of difference equations, the methods used and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. In fact, in the last 15 years, the proliferation of the subject has been witnessed by hundreds of research articles, several monographs, many international conferences and numerous special sessions.

In numerical integration of differential equations a standard approach is to replace it by a suitable difference equation whose solution can be obtained in a stable manner and without troubles from round-off errors. There are two types of solutions for difference equations, one is numerical (or summation form) another one is closed form (or exact solution). However, the qualitative properties of these solutions of the difference equations are quite different from the solutions of the corresponding differential equations. Solutions of several well known difference equations like Clairaut's, Euler's, Riccati's, Bernoulli's, Verhulst's, Duffing's, Mathieu's and Volterra's difference equations preserve most of the properties of the corresponding differential equations ([1]).

The basic theory of difference equations is based on the difference operator Δ defined as, $\Delta u(k) = u(k+1) - u(k)$, where $\{u(k)\}$ is a sequence of numbers. Many authors ([1],[20],[28]) have suggested the definition of generalized difference operator Δ_ℓ on $u(k)$, for real valued function defined on $\mathbb{R}$, as

$$\Delta_\ell u(k) = u(k+\ell) - u(k), \ k \in \mathbb{R}, \ \ell > 0. \tag{1.1}$$

E. Thandapani, M.Maria Susai Manuel, G.B.A Xavier [35] considered the definition of Δ_ℓ as given in (1.1) and developed the theory of difference equations in a different direction. If there exists a function $v(k)$ such that $\Delta_\ell v(k) = u(k)$, then we call this function $v(k)$ as $\Delta_\ell^{-1} u(k)$. In other words there exists a constant c_j such that

$$\Delta_\ell v(k) = u(k) \Rightarrow \ v(k) = \Delta_\ell^{-1} u(k) + c_j, \tag{1.2}$$

where c_j is constant for all $k \in \mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \ldots\}$, $j = k - [\frac{k}{\ell}]\ell$.

By defining the inverse Δ_ℓ^{-1}, many interesting results on sum of partial sums of higher power of arithmetic and geometric functions and applications in numerical methods ([35],[29]) have been obtained. The difference operator defined in (1.1) becomes the usual difference operator Δ when $\ell = 1$.

In 1989, Miller and Rose [30] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. In 1984, Jerzy Popenda [19] introduced a particular type of α-difference operator on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$, In 2011, M.Maria Susai Manuel, et.al, [27] extended the operator Δ_α to generalized $\alpha-$difference operator as $\underset{\alpha(\ell)}{\Delta} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function $v(k)$. In 2014, the authors in [8], have applied q-difference operator defined as $\Delta_q v(k) = v(qk) - v(k)$ and the difference operator $\underset{k(l)}{\Delta}$ with variable coefficients defined as $\underset{k(\ell)}{\Delta} v(k) = v(k+\ell) - kv(k)$.

These operators induce us to introduce the following partial difference operator. The generalized difference operator and its equation with n-shift values $l = (\ell_1, \ell_2, \ell_3, ..., \ell_n) \neq 0$ on a real valued function $v(k) : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\underset{(\ell)}{\Delta} v(k) \equiv v(k_1 + \ell_1, k_2 + \ell_2, ..., k_n + \ell_n) - v(k_1, k_2, ..., k_n) = u(k). \tag{1.3}$$

This operator $\underset{(\ell)}{\Delta}$ becomes generalized partial difference operator if some $\ell_i = 0$. The equations involving $\underset{(\ell)}{\Delta}$ with atleast one $\ell_i = 0$ is called generalized partial difference equation. A linear generalized partial difference equation $\underset{(\ell)}{\Delta} v(k) = u(k)$, has a numerical solution of the form,

$$v(k) - v(k - m\ell) = \sum_{r=1}^{m} u(k - r\ell) = \underset{(\ell)}{\overset{-1}{\Delta}} u(t)\Big|_{t=k-m\ell}^{k} \tag{1.4}$$

where $k - r\ell = (k_1 - r\ell_1, k_2 - r\ell_2, ..., k_n - r\ell_n)$ and m is any positive integer.

1.2 Discrete Heat Equation Model

This book aims to formulate and obtain numerical and exact (extorial) solutions of discrete partial heat equations of rod, thin plate and medium.

(a) Consider the temperature distribution of a very long rod. Assume that the rod is so long that it can be laid on top of the set $\Re$ of real numbers. Let $v(k_1, k_2)$ be the temperature at the real position k_1 and real time k_2 of the rod. Assume that diffusion rate γ is constant throughout the rod with shift values $\ell_i > 0$. By (1.3) and Fourier law of cooling, the discrete partial heat equation of the rod is formulated as

$$\underset{(0,\ell_2)}{\Delta} v(k_1, k_2) = \gamma \underset{(\pm\ell_1,0)}{\Delta} v(k_1, k_2), \tag{1.5}$$

where $\underset{(\pm\ell_1,0)}{\Delta} = \underset{(\ell_1,0)}{\Delta} + \underset{(-\ell_1,0)}{\Delta}$ and $\ell_1, \ell_2 > 0$.

(b) In the case of thin plate, let $v(k_1, k_2, k_3)$ be the temperature of the plate at position (k_1, k_2) and time k_3. The heat equation for the plate is

$$\underset{(0,0,\ell_3)}{\Delta} v(k) = \gamma \underset{\pm\ell_{(1,2)}}{\Delta} v(k), \tag{1.6}$$

where $\underset{(\pm\ell_{(1,2)})}{\Delta} = \underset{(\ell_1,0,0)}{\Delta} + \underset{(-\ell_1,0,0)}{\Delta} + \underset{(0,\ell_2,0)}{\Delta} + \underset{(0,-\ell_2,0)}{\Delta}$ and each $\ell_i > 0$.

(c) In the case of medium, let $v(k_1, k_2, k_3, k_4, k_5)$ be the temperature at position (k_1, k_2, k_3), at time k_4 with density (or pressure) k_5. By Fourier law of cooling, the heat equation for medium in $\Re^3$ is

$$\underset{(\ell_4,\ell_5)}{\Delta} v(k) = \gamma \underset{\pm\ell_{(1,2,3)}}{\Delta} v(k), \tag{1.7}$$

where $\underset{\pm\ell_{(1,2,3)}}{\Delta} = \underset{(\ell_1)}{\Delta} + \underset{(-\ell_1)}{\Delta} + \underset{(\ell_2)}{\Delta} + \underset{(-\ell_2)}{\Delta} + \underset{(\ell_3)}{\Delta} + \underset{(-\ell_3)}{\Delta}$ and $k = (k_1, k_2, k_3, k_4, k_5)$.

In this book, we derive numerical (summation) and exact (extorial) solutions of (1.5),(1.6) and (1.7), analyze some related Fibonacci fractional heat equations for rod, thin plate, medium and resistor-inductor (RL) circuit using MATLAB.

1.3 About the Book

This Book consists of seven chapters. The first chapter is devoted for necessary introduction to the thesis. The introduction contains of a description of the literature and developments in the field of difference equations.

The second chapter deals with the discrete heat equation model using the ℓ-difference and q-difference operators. Applying the Fourier law of cooling, the first breakthrough is done by the formulation of the discrete heat equation of the rod. The four types of numerical solutions for the heat equation formulation is derived.

Using a similar methodology, the solutions are also derived for plate and medium. The results thus obtained are analyzed for authenticity using MATLAB by assuming the boundary values. The diagrams generated help us to study the nature of the diffusion of heat within the material taken for study.

The third chapter provides with the formulation of delay heat equation model. The partial difference equation which extends its applications in heat equation is taken for study by the application of α-β difference operator and a model for heat transfer in the rod is arrived having recourse to Fourier law of cooling. The outcomes are eventually extended to thin plate and medium. The outcome obtained are validated by MATLAB. The outcomes arrived in this section gives the option for predicting the temperature by knowing the current values at the present time.

The fourth chapter focuses on solutions of partial difference equation with several variables. Being an application of difference operator, relevant formulae for finite and infinite series on polynomial and rational functions in number theory have been derived. This complex terrain of study finds its application in heat propagation within the given medium based on the Fourier law of conduction. It enables the optimal choice of material and gives us knowledge about the nature of the propagation of heat. The results are verified by MATLAB to validate the findings.

In the fifth chapter, partial Fibonacci difference equation is introduced and subjected to study in discrete heat equation by having recourse to Fibonacci difference operator with shift values. This investigation involves the definition of the Fibonacci difference operator, formulation of discrete heat equation model with several variables and the finding of the solutions of heat diffusion in materials with dimensions up to three. The partial Fibonacci difference operator provides a great possibility to study the various aspects of heat equation: the transfer of heat, nature of the material used and prediction of temperature with high accuracy having the knowledge of the present values as the basis. Simulations supported by MATLAB are inserted at relevant sections.

The sixth chapter analyses the flow of heat in a long rod made of non-homogeneous multiple materials stacked together is taken for study using partial difference equations with initial values assumed. With Newton's law of cooling as the basis, the Initial Value Problem (IVP) for heat transfer of the rod made of four non-homogeneous materials is formulated as the preliminary case. The solution arrived at for the IVP is generalized for the case of the rod with multiple materials. The results put forth in this book work are validated by numerical examples. The method presented here is very convenient for solving the heat equation and determining the temperature for all periods by having the knowledge of initial temperatures.

The seventh chapter presents the exact (closed form) solution of the heat equation model presented in the previous chapters. This investigation is made possible by the introduction of the new function which is defined by replacing polynomials into polynomial factorials in the expansion of exponential function denoted by $e_\nu(k_\ell)$ entitled as extorial function. This is an original and unique contribution to our book. Here, we focus on the fractional difference operator which plays a pivotal role in studying numerous systems and has been widely applied in various areas of study. As an application, we provide the solutions for discrete and fractional difference equations controlling current flows in RL circuit. The chaos created by the flow of current which generates heat energy in RL circuit is solved by applying the fractional difference equation.

Chapter 2

Discrete Heat Equation Model with Several Variables

2.1 Introduction

In this chapter, we investigate the generalized partial difference operator and propose a model of it in discrete heat equation. The diffusion of heat is studied by the application of Newton's law of cooling in dimensions up to three and several solutions are postulated for the same. Through numerical simulations using MATLAB, solutions are validated and applications are derived.

2.2 Discrete heat equation of rod

Consider the temperature distribution of a very long rod, notations, discrete heat equation (1.5) mentioned for the research problem in section 1.2(a). Since the unknown function $v(k_1, k_2)$ lies on both sides of equation (1.5), finding exact solution is a challenge one. We have overcome this problem. Here, we derive the temperature formula for $v(k_1, k_2)$ at the general position (k_1, k_2).

Theorem 2.2.1. *Assume that there exists a positive integer m, and a real number $\ell_2 > 0$ such that $v(k_1, k_2 - m\ell_2)$ and $\underset{\pm\ell_1}{\Delta} v(k_1, k_2) = \underset{\pm\ell_1}{u}(k_1, k_2)$ are known. Then, the heat equation (1.5) has a solution $v(k_1, k_2)$ of the form*

$$v(k_1, k_2) = v(k_1, k_2 - m\ell_2) + \gamma \sum_{r=1}^{m} \underset{\pm\ell_1}{u}(k_1, k_2 - r\ell_2). \tag{2.1}$$

Proof. Taking $\underset{(\pm\ell_1,0)}{\Delta} v(k_1, k_2) = \underset{\pm\ell_1}{u}(k_1, k_2)$ in (1.5) gives

$$v(k_1, k_2) = \gamma \underset{(0,\ell_2)}{\Delta^{-1}} \underset{\pm\ell_1}{u}(k_1, k_2). \tag{2.2}$$

The proof of (2.1) follows by applying the inverse principle (1.4) in (2.2). □

Example 2.2.2. *From (1.3), we get,* $\underset{(0,\ell_2)}{\overset{-1}{\Delta}} e^{i(k_1+k_2)} = \dfrac{e^{i(k_1+k_2-\ell_2)} - e^{i(k_1+k_2)}}{2(1-\cos\ell_2)}$, *whose imaginary part yields*

$$\underset{(0,\ell_2)}{\overset{-1}{\Delta}} \sin(k_1+k_2) = \frac{\sin(k_1+k_2-\ell_2) - \sin(k_1+k_2)}{2(1-\cos\ell_2)}. \tag{2.3}$$

Taking $\underset{\pm\ell_1}{u}(k_1, k_2) = \sin(k_1+k_2+\ell_1) - \sin(k_1+k_2-\ell_1)$ *in (2.2), using (2.3), (2.1),*

$$\frac{\underset{\pm\ell_1}{u}(k_1,k_2-\ell_2)-\underset{\pm\ell_1}{u}(k_1,k_2)}{2(1-\cos\ell_2)}=\frac{\underset{\pm\ell_1}{u}(k_1,k_2-(m+1)\ell_2)-\underset{\pm\ell_1}{u}(k_1,k_2-m\ell_2)}{2(1-\cos\ell_2)}$$
$$+\sum_{r=1}^{m}\underset{\pm\ell_1}{u}(k_1,k_2-r\ell_2). \tag{2.4}$$

The MATLAB coding of (2.4) for $m=50,\ k_1=2,\ \ell_1=3,\ k_2=4,\ \ell_2=5$ is below: $(sin(4)-sin(9)+sin(-2)-sin(3))./(2.*(1-cos(5)))=(sin(-246)-sin(-241)+sin(-252)-sin(-247))./(2.*(1-cos(5)))+symsum(sin(9-5.*r)+sin(3-5.*r),r,1,50)$.

Theorem 2.2.3. *Consider (1.5) and denote $v(k_1\pm\ell_1,*)=v(k_1+\ell_1,*)+v(k_1-\ell_1,*)$ and $v(*,k_2\pm\ell_2)=v(*,k_2+\ell_2)+v(*,k_2-\ell_2)$. Then, the following four type solutions of the equation (1.5) are equivalent:*

(a). $v(k_1,k_2)=(1-2\gamma)^m v(k_1,k_2-m\ell_2)+\sum_{r=0}^{m-1}\gamma(1-2\gamma)^r\big[v(k_1\pm\ell_1,k_2-(r+1)\ell_2)\big],$ (2.5)

(b). $v(k_1,k_2)=\dfrac{1}{(1-2\gamma)^m}v(k_1,k_2+m\ell_2)-\sum_{r=1}^{m}\dfrac{\gamma}{(1-2\gamma)^r}[v(k_1\pm\ell_1,k_2+(r-1)\ell_2)],$ (2.6)

(c). $v(k_1,k_2)=\dfrac{1}{\gamma^m}v(k_1-m\ell_1,k_2+m\ell_2)-\sum_{r=1}^{m}\dfrac{1-2\gamma}{\gamma^r}[v(k_1-r\ell_1,k_2+(r-1)\ell_2)]$

$$-\sum_{s=0}^{m-1}\frac{1}{\gamma^s}[v(k_1-(s+2)\ell_1,k_2+s\ell_2)], \tag{2.7}$$

(d). $v(k_1,k_2)=\dfrac{1}{\gamma^m}v(k_1+m\ell_1,k_2+m\ell_2)-\sum_{r=1}^{m}\dfrac{1-2\gamma}{\gamma^r}[v(k_1+r\ell_1,k_2+(r-1)\ell_2)]$

$$-\sum_{s=0}^{m-1}\frac{1}{\gamma^s}[v(k_1+(s+2)\ell_1,k_2+s\ell_2)]. \tag{2.8}$$

Proof. (a). From (1.5), we arrive the relation

$$v(k_1, k_2) = (1 - 2\gamma)v(k_1, k_2 - \ell_2) + \gamma v(k_1 \pm \ell_1, k_2 - \ell_2). \tag{2.9}$$

Replacing k_2 by $k_2 - r\ell_2$ in (2.9) gives expressions for $v(k_1, k_2 - r\ell_2)$ and $v(k_1 \pm \ell_1, k_2 - r\ell_2)$ for $r = 1, 2, ..., m$. Now proof of (a) follows by applying all these values in (2.9).

(b). The heat equation (1.5) directly yields the relation

$$v(k_1, k_2) = \frac{1}{(1 - 2\gamma)} v(k_1, k_2 + \ell_2) - \frac{\gamma}{(1 - 2\gamma)} v(k_1 \pm \ell_1, k_2). \tag{2.10}$$

Replacing k_2 by $k_2 + r\ell_2$ and substituting corresponding γ-values in (2.10) gives (b).

(c). The proof of (c) follows by replacing k_1 by $k_1 - r\ell_1$ and k_2 by $k_2 + r\ell_2$ and $v(k_1, k_2) = \dfrac{1}{\gamma} v(k_1 - \ell_1, k_2 + \ell_2) - \dfrac{1 - 2\gamma}{\gamma} v(k_1 - \ell_1, k_2) - v(k_1 - 2\ell_1, k_2)$.

(d). The proof of (d) follows by replacing k_1 by $k_1 + r\ell_1$ and k_2 by $k_2 + r\ell_2$ and $v(k_1, k_2) = \dfrac{1}{\gamma} v(k_1 + \ell_1, k_2 + \ell_2) - \dfrac{1 - 2\gamma}{\gamma} v(k_1 + \ell_1, k_2) - v(k_1 + 2\ell_1, k_2)$. □

Example 2.2.4. *The following example shows that the diffusion rate of rod can be identified if the solution $v(k_1, k_2)$ of (1.5) is known and vice versa.*

Suppose that $v(k_1, k_2) = a^{k_1+k_2}$ is a closed form solution of (1.5), then we have

$$\underset{(0,\ell_2)}{\Delta} a^{k_1+k_2} = \gamma \big[\underset{(\ell_1,0)}{\Delta} a^{k_1+k_2} + \underset{(-\ell_1,0)}{\Delta} a^{k_1+k_2} \big],$$

which yields $a^{k_1+k_2+\ell_2} - a^{k_1+k_2} = \gamma \big[a^{k_1+k_2+\ell_1} + a^{k_1+k_2-\ell_1} - 2a^{k_1+k_2} \big]$.

Cancelling $a^{k_1+k_2}$ on both sides derives $\gamma = \dfrac{a^{\ell_2} - 1}{a^{\ell_1} + a^{-\ell_1} - 2}$.

Theorem 2.2.5. *Assume that the heat difference* $\underset{(-\ell_1,0)}{\Delta} v(k_1,k_2)$ *is proportional to* $\underset{(\ell_1,0)}{\Delta} v(k_1,k_2)$ *i.e.,* $\underset{(-\ell_1,0)}{\Delta} v(k_1,k_2) = \delta \underset{(\ell_1,0)}{\Delta} v(k_1,k_2)$. *In this case, the heat equation* (1.5) *has a solution* $\cos(k_1+k_2)$ *if and only if either* $\cos(k_1+k_2)=0$ *or* $\sin\ell_1 = 0$.

Proof. From the heat equation (1.5) and the given condition we arrive

$$\underset{(0,\ell_2)}{\Delta} v(k_1,k_2) = \gamma(1+\delta) \underset{(\ell_1,0)}{\Delta} v(k_1,k_2). \tag{2.11}$$

If, $\cos(k_1+k_2) = \dfrac{\left[e^{i(k_1+k_2)} + e^{-i(k_1+k_2)}\right]}{2} = v(k_1,k_2)$, then (2.11) becomes

$\underset{(0,\ell_2)}{\Delta} \left[e^{i(k_1+k_2)} + e^{-i(k_1+k_2)}\right] = \gamma(1+\delta) \underset{(\ell_1,0)}{\Delta} \left[e^{i(k_1+k_2)} + e^{-i(k_1+k_2)}\right]$,

which yields $e^{i(k_1+k_2+l_2)} + e^{-i(k_1+k_2+l_2)} - e^{i(k_1+k_2)} - e^{-i(k_1+k_2)}$

$$= \gamma(1+\delta)e^{i(k_1+k_2+l_1)} + e^{-i(k_1+k_2+l_1)} - e^{i(k_1+k_2)} - e^{-i(k_1+k_2)}.$$

By rearranging the terms, we find

$e^{i(k_1+k_2)}\left[e^{il_2} - 1 - \gamma(1+\delta)e^{i\ell_1} - 1\right] = e^{-i(k_1+k_2)}\left[e^{i\ell_2} - 1 - \gamma(1+\delta)e^{-i\ell_1} - 1\right]$,

which yields either $e^{i(k_1+k_2)} + e^{-i(k_1+k_2)} = 0$ or $e^{i\ell_1} = e^{-i\ell_2}$.

Hence $\cos(k_1+k_2) = 0$ or $\sin l_1 = 0$. Retracing the steps gives converse. $\square$

2.3 Solution of heat equation for thin plate

The discrete heat equation for the thin plate in section 1.2(b) is given by the equation

$$\underset{(0,0,\ell_3)}{\Delta} v(k) = \gamma \underset{\pm\ell_{(1,2)}}{\Delta} v(k), \underset{\pm\ell_{(1,2)}}{\Delta} = \underset{(\ell_1,0,0)}{\Delta} + \underset{(-\ell_1,0,0)}{\Delta} + \underset{(0,\ell_2,0)}{\Delta} + \underset{(0,-\ell_2,0)}{\Delta}. \tag{2.12}$$

Here we obtain four types of solutions of (2.12) as in the rod case.

Theorem 2.3.1. *Consider the discrete heat equation (2.12). Assume that there exists a positive integer m, and a real number $\ell_3 > 0$ such that $v(k_1, k_2, k_3 - ml_3)$ and the partial differences $\underset{\pm\ell_{(1,2)}}{\Delta} v(k_1, k_2, k_3) = \underset{\pm\ell_{(1,2)}}{u}(k_1, k_2, k_3)$ are known functions. Then, the heat equation (2.12) has a solution $v(k_1, k_2, k_3)$ as*

$$v(k) = v(k_1, k_2, k_3 - m\ell_3) + \gamma \sum_{r=1}^{m} \underset{\pm\ell_{(1,2)}}{u}(k_1, k_2, k_3 - r\ell_3). \tag{2.13}$$

Proof. Taking $\underset{\pm\ell_{(1,2)}}{\Delta} v(k) = \underset{\pm\ell_{(1,2)}}{u}(k)$ in (2.12), we find

$$v(k) = \gamma \underset{(0,0,\ell_3)}{\Delta^{-1}} \underset{\pm\ell_{(1,2)}}{u}(k). \tag{2.14}$$

The proof of (2.13) follows by applying inverse principle of $\overset{-1}{\underset{\ell_3}{\Delta}}$ on (2.14). □

Consider the following notations which will be used in the Theorem 2.3.2.

$$v(k_{(1,2)} \pm \ell_{(1,2)}, *) = v(k_1 \pm \ell_1, k_2, *) + v(k_1, k_2 \pm \ell_2, *) \text{ also}$$

$$v(*, k_{(2,3)} \pm \ell_{(2,3)}) = v(*, k_2 \pm \ell_2, k_3) + v(*, k_2, k_3 \pm \ell_3).$$

Theorem 2.3.2. *Assume that $v(k_1, k_2, k_3)$ is a solution of equation (2.12), $v(k_1 \pm r\ell_1, k_2 \pm r\ell_2)$ exists and denote $v(k_1 \pm \ell_1, *, *) = v(k_1 + \ell_1, *, *) + v(k_1 - \ell_1, *, *)$, $v(*, *, k_3 \pm \ell_3) = v(*, *, k_3 + \ell_3) + v(*, *, k_3 - \ell_3)$. Then, the following are equivalent:*

(a). $v(k_1, k_2, k_3) = (1 - 4\gamma)^m v(k_1, k_2, k_3 - m\ell_3) + \sum\limits_{r=0}^{m-1} \gamma(1 - 4\gamma)^r \times$

$$\big[v(k_1 \pm \ell_1, k_2, k_3 - (r+1)\ell_3) + v(k_1, k_2 \pm \ell_2, k_3 - (r+1)\ell_3)\big], \tag{2.15}$$

(b). $v(k_1, k_2, k_3) = \dfrac{1}{(1 - 4\gamma)^m} v(k_1, k_2, k_3 + m\ell_3)$

$$- \sum_{r=1}^{m} \frac{\gamma}{(1 - 4\gamma)^r} \big[v(k_{(1,2)} \pm l_{(1,2)}, k_3 + (r-1)\ell_3)\big], \tag{2.16}$$

(c). $v(k) = \dfrac{1}{\gamma^m} v(k_1 - m\ell_1, k_2, k_3 + m\ell_3) - \sum_{r=1}^{m} \dfrac{1-4\gamma}{\gamma^r}[v(k_1 - r\ell_1, k_2, k_3 + (r-1)\ell_3)]$

$$- \sum_{r=0}^{m-1} \frac{1}{\gamma^r}[v(k_1 - (r+2)\ell_1, k_2, k_3 + r\ell_3)] - \sum_{r=0}^{m-1} \frac{1}{\gamma^r}[v(k_1 - (r+1)\ell_1, k_2 \pm \ell_2, k_3 + r\ell_3)], \tag{2.17}$$

(d). $v(k) = \dfrac{1}{\gamma^m} v(k_1 + m\ell_1, k_2, k_3 + m\ell_3) - \sum_{r=1}^{m} \dfrac{1-4\gamma}{\gamma^r}[v(k_1 + r\ell_1, k_2, k_3 + (r-1)\ell_3)]$

$$- \sum_{r=0}^{m-1} \frac{1}{\gamma^r}[v(k_1 + (r+2)\ell_1, k_2, k_3 + r\ell_3)] - \sum_{r=0}^{m-1} \frac{1}{\gamma^r}[v(k_1 + (r+1)\ell_1, k_2 \pm \ell_2, k_3 + r\ell_3)]. \tag{2.18}$$

Proof. From (2.12) and (1.3), we arrive

(i). $v(k) = (1-4\gamma)v(k_1, k_2, k_3 - \ell_3) + \gamma[v(k_1 \pm \ell_1, k_2, k_3 - \ell_3) + v(k_1, k_2 \pm \ell_2, k_3 - \ell_3)]$.

(ii). $v(k) = \dfrac{1}{(1-4\gamma)} v(k_1, k_2, k_3 + \ell_3) - \dfrac{\gamma}{(1-4\gamma)}[v(k_1 \pm \ell_1, k_2, k_3) + v(k_1, k_2 \pm \ell_2, k_3)]$.

(iii). $v(k) = \dfrac{1}{\gamma} v(k_1 - \ell_1, k_2, k_3 + \ell_3) - \dfrac{1-4\gamma}{\gamma} v(k_1 - \ell_1, k_2, k_3)$

$$- v(k_1 - 2\ell_1, k_2, k_3) - v(k_1 - \ell_1, k_2 \pm \ell_2, k_3).$$

(iv). $v(k) = \dfrac{1}{\gamma} v(k_1 + \ell_1, k_2, k_3 + \ell_3) - \dfrac{1-4\gamma}{\gamma} v(k_1 + \ell_1, k_2, k_3)$

$$- v(k_1 + 2\ell_1, k_2, k_3) - v(k_1 + \ell_1, k_2 \pm \ell_2, k_3).$$

Now the proof of (a), (b), (c), (d) follows by replacing

k_3 by $k_3 - \ell_3, k_3 - 2\ell_3, ..., k_m - m\ell_3$ in (i) k_3 by $k_3 + \ell_3, k_3 + 2\ell_3, ..., k_m + m\ell_3$ in (ii)

k_1 and k_3 by $k_1 - \ell_1, k_1 - 2\ell_1, ..., k_m - m\ell_1,\ k_3 + \ell_3, k_3 + 2\ell_3, ..., k_m + m\ell_3$ in (iii)

k_1 and k_3 by $k_1 + \ell_1, k_1 + 2\ell_1, ..., k_m + m\ell_1,\ k_3 + \ell_3, k_3 + 2\ell_3, ..., k_m + m\ell_3$ in (iv)

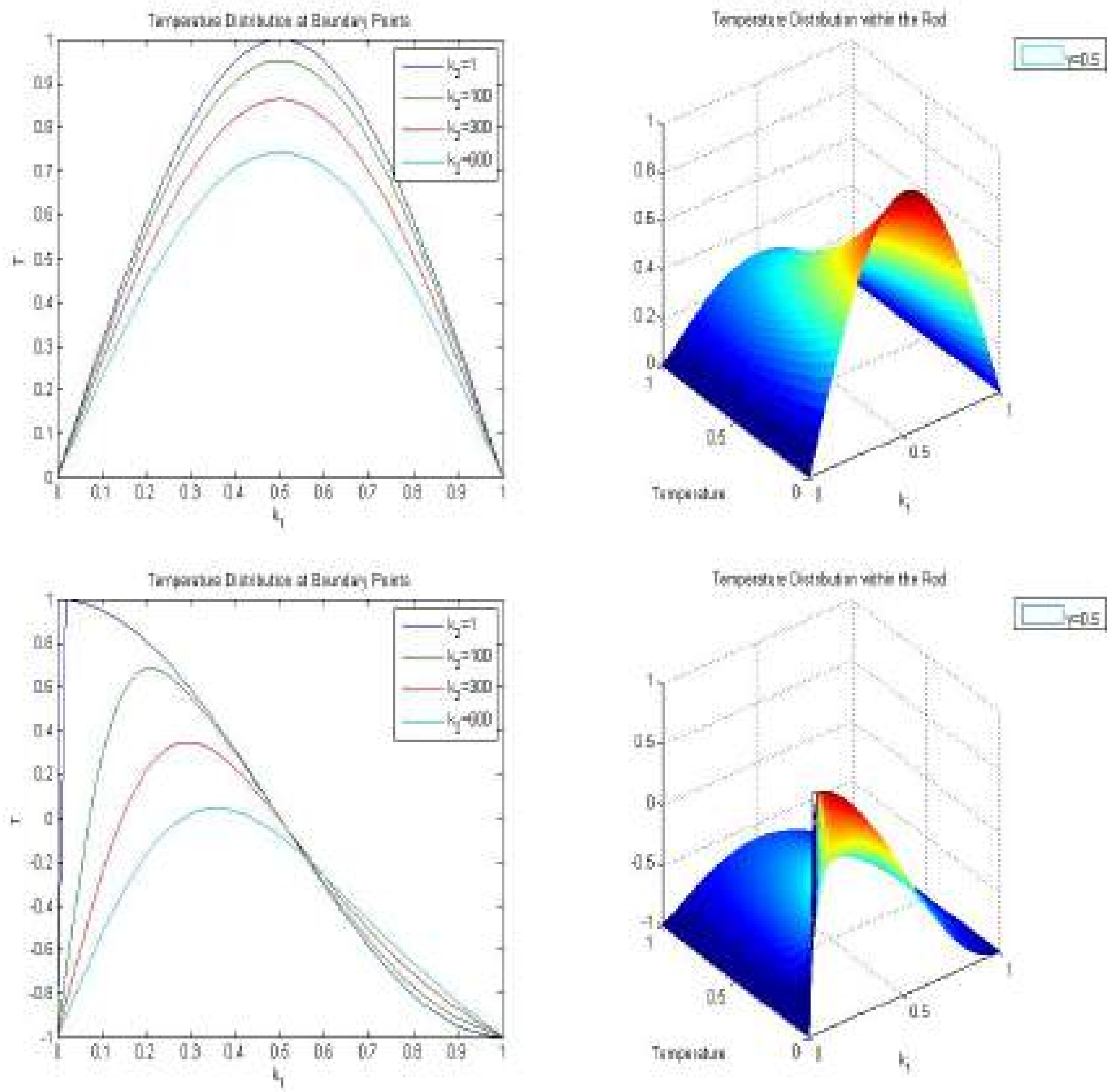
respectively. □

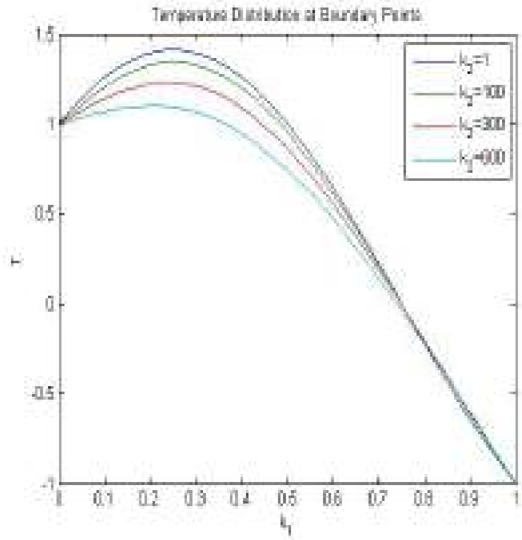
The following diagrams are obtained by using (2.9), by taking $\gamma = 0.5$, $\ell_1 = \dfrac{1}{50}$ and $\ell_2 = \dfrac{1}{2500}$, (i). boundary values for sine function are $v(k_1{,}1) = \sin \pi \ell_1$, $v(1{,}k_2) = 0, v(51{,}k_2) = 0$,

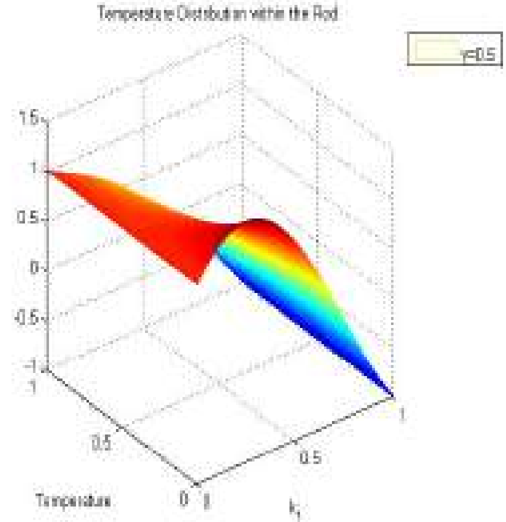
(ii). for cosine function, $v(k_1, 1) = \cos \pi \ell_1$, $v(1, k_2) = -1$, $v(51, k_2) = -1$,

(iii). for sum of sine and cosine function, $v(k_1, 1) = \sin \pi \ell_1 + \cos \pi \ell_1$,

$v(1, k_2) = 1, v(51, k_2) = -1$ respectively.







From the above diagrams, when the transmission of heat is known at the boundary points then the diffusion within the material under study can be easily determined.

2.4 Solution of heat equation for medium

Consider the research problem for medium mentioned in section 1.2(c). Let γ be heat diffusion constant, the proportional amount of heat flows from left to right at $(k_1, k_2, k_3, k_4, k_5)$ is $\underset{(-\ell_1,0,0)}{\Delta} v(k)$, right to left $\underset{(\ell_1,0,0)}{\Delta} v(k)$, top to bottom $\underset{(0,\ell_2,0)}{\Delta} v(k)$, bottom to top $\underset{(0,-\ell_2,0)}{\Delta} v(k)$, front to rear $\underset{(0,0,\ell_3)}{\Delta} v(k)$, rear to front $\underset{(0,0,-\ell_3)}{\Delta} v(k)$. By Fourier law of cooling, the heat equation for medium in $\Re^3$ is given by

$$\underset{(\ell_4,\ell_5)}{\Delta} v(k) = \gamma \underset{\pm\ell_{(1,2,3)}}{\Delta} v(k), \tag{2.19}$$

where $\underset{\pm\ell_{(1,2,3)}}{\Delta} = \underset{(\ell_1)}{\Delta} + \underset{(-\ell_1)}{\Delta} + \underset{(\ell_2)}{\Delta} + \underset{(-\ell_2)}{\Delta} + \underset{(\ell_3)}{\Delta} + \underset{(-\ell_3)}{\Delta}$ and $k = (k_1, k_2, k_3, k_4, k_5)$.

Theorem 2.4.1. *Assume that* $v(k_1, k_2, k_3, k_4 - m\ell_4, k_4 - m\ell_5)$ *and the partial differences* $\underset{\pm\ell_{(1,2,3)}}{\Delta} v(k) = \underset{\pm\ell(1,2,3)}{u}(k)$ *are known functions. Then, the heat equation* (2.19) *has a solution of the form,* $v(k) = v(k_1, k_2, k_3, k_4 - m\ell_4, k_5 - m\ell_5)$

$$+ \gamma \sum_{r=1}^{m} \underset{\pm\ell_{(1,2,3)}}{u}(k_1, k_2, k_3, k_4 - r\ell_4, k_5 - m\ell_5). \tag{2.20}$$

Proof. Taking $\underset{\pm\ell_{(1,2,3)}}{\Delta} v(k) = \underset{\pm\ell_{(1,2,3)}}{u}(k)$ in (2.19), we get

$$v(k) = \gamma \underset{(\ell_4, \ell_5)}{\Delta^{-1}} \underset{(\pm\ell_{(1,2,3)})}{u}(k). \tag{2.21}$$

The proof follows by applying inverse principle (2.1) on (2.21). □

In the Theorem 2.4.2, we use the following notations:

$$\begin{aligned} v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, *, *) &= v(k_1 + \ell_1, k_2, k_3, *, *) + v(k_1 - \ell_1, k_2, k_3, *, *) \\ &\quad + v(k_1, k_2 + \ell_2, k_3, *, *) + v(k_1, k_2 - \ell_2, k_3, *, *) \\ &\quad + v(k_1, k_2, k_3 + \ell_3, *, *) + v(k_1, k_2, k_3 - \ell_3, *, *). \end{aligned}$$

$$\begin{aligned} v(*, k_{(2,3)} \pm \ell_{(2,3)}, *, *) &= v(*, k_2 + \ell_2, k_3, *, *) + v(*, k_2 - \ell_2, k_3, *, *) \\ &\quad + v(*, k_2, k_3 + \ell_3, *, *) + v(*, k_2, k_3 - \ell_3, *, *). \end{aligned}$$

Theorem 2.4.2. *If* $v(k)$ *is a solution of the equation* (2.19) *and* m *is a positive integer then the following relations are equivalent:*

(a) $v(k) = (1 - 6\gamma)^m v(k_1, k_2, k_3, k_4 - m\ell_4, k_5 - m\ell_5)$

$$+ \sum_{r=0}^{m-1} \gamma(1 - 6\gamma)^r \Big[v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4 - (r+1)\ell_4, k_5 - (r+1)\ell_5) \Big], \tag{2.22}$$

(b) $v(k) = \frac{1}{(1-6\gamma)^m} v(k_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5)$

$$- \sum_{r=1}^{m} \frac{\gamma}{(1-6\gamma)^r} \Big[v(k_{(1,2,3)} \pm l_{(1,2,3)}, k_4 + (r-1)\ell_4, k_5 + (r-1)\ell_5) \Big], \tag{2.23}$$

(c) $v(k) = \frac{1}{\gamma^m} v(k_1 - m\ell_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5)$

$$- \sum_{r=1}^{m} \frac{1-6\gamma}{\gamma^r} [v(k_1 - r\ell_1, k_2, k_3, k_4 + (r-1)\ell_4, k_5 + (r-1)\ell_5)]$$

$$- \sum_{r=0}^{m-1} \frac{1}{\gamma^r} [v(k_1 - (r+1)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5)], \tag{2.24}$$

(d) $v(k) = \frac{1}{\gamma^m} v(k_1 + m\ell_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5)$

$$- \sum_{r=1}^{m} \frac{1-6\gamma}{\gamma^r} [v(k_1 + r\ell_1, k_2, k_3, k_4 + (r-1)\ell_4, k_5 + (r-1)\ell_5)]$$

$$- \sum_{r=0}^{m-1} \frac{1}{\gamma^r} [v(k_1 + (r+1)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5)]. \tag{2.25}$$

Proof. From (2.19) and (1.3), we arrive

(i) $v(k) = (1-6\gamma)v(k_1, k_2, k_3, k_4 - \ell_4, k_5 - \ell_5) + \gamma[v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4 - \ell_4, k_5 - \ell_5)$,

(ii) $v(k) = \frac{1}{(1-6\gamma)} v(k_1, k_2, k_3, k_4 + \ell_4, k_5 + \ell_5) - \frac{\gamma}{(1-6\gamma)} [v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4, k_5)]$,

(iii) $v(k) = \frac{1}{\gamma} v(k_1 - \ell_1, k_2, k_3, k_4 + \ell_4, k_5 + \ell_5) - \frac{1-6\gamma}{\gamma} v(k_1 - \ell_1, k_2, k_3, k_4, k_5)$

$- v(k_1 - 2\ell_1, k_2, k_3, k_4, k_5) - v(k_1 - \ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4, k_5)$ and

(iv) $v(k) = \frac{1}{\gamma} v(k_1 + \ell_1, k_2, k_3, k_4 + \ell_4, k_5 + \ell_5) - \frac{1-6\gamma}{\gamma} v(k_1 + \ell_1, k_2, k_3, k_4, k_5)$

$- v(k_1 + 2\ell_1, k_2, k_3, k_4, k_5) - v(k_1 + \ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4, k_5)$.

Now the proofs of (a), (b), (c) and (d) follow by replacing

k_4 and k_5 by $k_4 - \ell_4, k_4 - 2\ell_4, ..., k_m - m\ell_4,\ k_5 - \ell_5, k_5 - 2\ell_5, ..., k_m - m\ell_5,$

k_4 and k_5 by $k_4+\ell_4, k_4+2\ell_4, ..., k_m+m\ell_4$, $k_5+\ell_5, k_5+2\ell_5, ..., k_m-m\ell_5$,

k_1 by $k_1-\ell_1, k_1-2\ell_1, ..., k_m-m\ell_1$, k_4 by $k_4+\ell_4, k_4+2\ell_4, ..., k_m+m\ell_4$ and

k_5 by $k_5+\ell_5, k_5+2\ell_5, ..., k_m-m\ell_5$,

k_1 by $k_1+\ell_1, k_1+2\ell_1, ..., k_m+m\ell_1$, k_4 by $k_4+\ell_4, k_4+2\ell_4, ..., k_m+m\ell_4$ and

k_5 by $k_5+\ell_5, k_5+2\ell_5, ..., k_m-m\ell_5$ in (i), (ii), (iii) and (iv) respectively. $\square$

The following example shows that the diffusion of medium in three dimensional system can be identified if the solution $v(k_1,k_2,k_3,k_4,k_5)$ of (2.19) is known.

Example 2.4.3. *Suppose that* $v(k_1,k_2,k_3,k_4,k_5)=e^{k_1+k_2+k_3+k_4+k_5}$ *is a closed form solution of (2.19), then we have* $\underset{(\ell_4,\ell_5)}{\Delta} e^{k_1+k_2+k_3+k_4+k_5}=\gamma\big[\underset{\pm\ell_{(1,2,3)}}{\Delta} e^{k_1+k_2+k_3+k_4+k_5}\big]$*, which yields,* $e^{k_1+k_2+k_3+k_4+k_5}(e^{\ell_4+\ell_5}-1)=\gamma\big[e^{k_1+k_2+k_3+k_4+k_5}(e^{\ell_1}+e^{-\ell_1}+e^{\ell_2}+e^{-\ell_2}+e^{\ell_3}+e^{-\ell_3}-6\big]$*. Cancelling* $e^{k_1+k_2+k_3+k_4+k_5}$ *on both sides derives*

$$\gamma=\frac{e^{\ell_4+\ell_5}-1}{e^{\ell_1}+e^{-\ell_1}+e^{\ell_2}+e^{-\ell_2}+e^{\ell_3}+e^{-\ell_3}-6}. \tag{2.26}$$

For the numerical verification of Theorem 2.4.2 (a), if we take $k_1=1,k_2=2,k_3=3$, $k_4=4,k_5=5,\ell_1=1,\ell_2=2,\ell_3=3,\ell_4=4,\ell_5=5,m=1$*, then* $v(k_1,k_2,k_3,k_4,k_5)=e^{15}$ *and* $\gamma=\dfrac{e^9-1}{e^1+e^{-1}+e^2+e^{-2}+e^3+e^{-3}-6}$.

The corresponding MATLAB coding is given below:

$exp(15)=(1-6.*(327.4114733)).\wedge(5).*exp(-30)+symsum((327.4114733).*(1-6.*(327.4114733)).\wedge r.*((exp(16-(r+1).*4-(r+1).*5))+(exp(14-(r+1).*4-(r+1).*5))+(exp(17-(r+1).*4-(r+1).*5))+(exp(13-(r+1).*4-(r+1).*5))+(exp(18-(r+1).*4-(r+1).*5))+(exp(12-(r+1).*4-(r+1).*5))),r,0,4)$.

Similarly, we can easily verify the results for Theorem 2.4.2(b),(c) and (d).

The heat equation arrived by ℓ-difference equations can be extended to q-difference equations

2.5 Discrete q-heat equation

Consider, the two side temperature distribution of a very long rod. Let $v(k_1, k_2)$ be the temperature at the real time k_2 and real position k_1 of the rod. At time k_2, if the temperature $v\Big(\dfrac{k_1}{q_1}, k_2\Big), q_1 > 0$ is higher than $v(k_1, k_2)$, heat will flow from the point $\dfrac{k_1}{q_1}$ to k_1. Similarly, at time k_2, if the temperature $v(k_1q_1, k_2), q_1 > 0$ is higher than $v(k_1, k_2)$, heat will flow from the point k_1q_1 to k_1.

The amount of increase is $v(k_1, k_2q_2) - v(k_1, k_2)$ is proportional to the difference, $v\Big(\dfrac{k_1}{q_1}, k_2\Big) - v(k_1, k_2)$ say, $\alpha(k_1, k_2)\left(v\Big(\dfrac{k_1}{q_1}, k_2\Big) - v(k_1, k_2)\right)$ and the amount of increase is $v(k_1, k_2q_2) - v(k_1, k_2)$ is proportional to the difference, $v(k_1q_1, k_2) - v(k_1, k_2)$, $\alpha\left(v(k_1q_1, k_2) - v(k_1, k_2)\right)$, α-positive diffusion rate constant $v(k_1, k_2q_2) - v(k_1, k_2) = \alpha\left(v(\dfrac{k_1}{q_1}, k_2) - v(k_1, k_2)\right) + \alpha(v(k_1q_1, k_2) - v(k_1, k_2)), \alpha > 0$. Here 1-D denotes one dimension.

Definition 2.5.1. *An 1-D two side q-heat equation is defined as*

$$\underset{q_2}{\Delta} v(k_1, k_2) = \alpha \underset{q_1^{-1}}{\Delta} v(k_1, k_2) + \alpha \underset{q_1}{\Delta} v(k_1, k_2). \tag{2.27}$$

Similarly, 1-D two side q-heat equation with variable coefficient is defined as

$$\underset{q_2}{\Delta} v(k_1, k_2) = \alpha(k_2, k_1) \underset{q_1^{-1}}{\Delta} v(k_1, k_2) + \alpha(k_2, k_1) \underset{q_1}{\Delta} v(k_1, k_2), \tag{2.28}$$

where $\alpha(k_2, k_1)$ is a function of k_2 and k_1.

Remark 2.5.2. *If $v(k_1, k_2) = k_2k_1$ is a solution of (2.28), then*

$$\alpha(k_2, k_1) = \frac{(q_2-1)q_1}{(1-q_1)^2}, \text{ which is a constant.}$$

2.6 Two side flow q-heat equation with constant coefficient

In this section, we derive a solution of discrete two side flow q-heat equation (2.27) and also we obtain a function $v(k_1, k_2)$ satisfying the equation (2.27).

Theorem 2.6.1. *Four types of solutions of 1-D two side q-heat equation (2.27) are*

$$(i)\ v(k_1,k_2) - v\Big(k_1, \frac{k_2}{q_2^m}\Big) = \alpha \sum_{r=1}^{m}\left[v\Big(\frac{k_1}{q_1}, k_2q_2^{-r}\Big) + v(k_1q_1, k_2q_2^{-r}) - 2v(k_1,k_2)\right], \tag{2.29}$$

$$(ii)\ v(k_1,k_2) = \frac{1}{(1-\alpha)^m}v(k_1,k_2q_2^m) - \sum_{r=1}^{m}\frac{\alpha}{(1-\alpha)^r}\left(v\Big(\frac{k_1}{q_1}, k_2q_2^{r-1}\Big) + v(k_1q_1, k_2q_2^{r-1})\right), \tag{2.30}$$

$$(iii)\ v(k_1,k_2) = \frac{1}{1-2\alpha}v(k_1,k_2q_2) - \frac{\alpha}{1-2\alpha}v(k_1q_1,k_2) - \frac{\alpha}{(1-2\alpha)^{m+1}}v\Big(\frac{k_1}{q_1}, k_2q_2^m\Big) + \sum_{r=1}^{m}\frac{\alpha^2}{(1-2\alpha)^{r+1}}\left\{v\Big(\frac{k_1}{q_1^2}, k_2q_2^{(r-1)}\Big) + v(k_1, k_2q_2^{(r-1)})\right\}, \tag{2.31}$$

$$(iv)\ v(k_1,k_2) = \frac{1}{1-2\alpha}v(k_1,k_2q_2) - \frac{\alpha}{1-2\alpha}v\Big(\frac{k_1}{q_1},k_2\Big) - \frac{\alpha}{(1-2\alpha)^{m+1}}v(k_1q_1, k_2q_2^m) + \sum_{r=1}^{m}\frac{\alpha^2}{(1-2\alpha)^{r+1}}\left\{v(k_1, k_2q_2^{(r-1)}) + v(k_1q_1^2, k_2q_2^{(r-1)})\right\}. \tag{2.32}$$

Proof. (i) From the linearity of $\overset{-1}{\underset{q_2,q_1}{\Delta}}$ and (2.27), we have

$$v(k_1,k_2)=\alpha\overset{-1}{\underset{q_2}{\Delta}}\left(\underset{q_1{}^{-1}}{\Delta}v(k_1,k_2)+\underset{q_1}{\Delta}v(k_1,k_2)\right). \tag{2.33}$$

$$\underset{q_1^{-1}}{\Delta}v(k_1,k_2)=u(k_1,k_2) \text{ and } \underset{q_1}{\Delta}v(k_1,k_2)=w(k_1,k_2), \tag{2.34}$$

are known functions, then we have a solution to (2.27) as

$$v(k_1,k_2)-v(k_1,\frac{k_2}{q_2^m})=\alpha\sum_{r=1}^{m}\left\{u(k_2q_2^{-r},k_1)+w(k_2q_2^{-r},k_1)\right\}. \tag{2.35}$$

In (2.35), $u(k_2q_2^{-r_1},k_1)$ is obtained by replacing k_2 by $k_2q_2^{-r_1}$ in (2.34) .

Substituting (2.34) in (2.35), we get (2.29) which is first type solution of (2.27).

(ii) From the q-difference heat equation (2.27), we arrive

$$v(k_1,k_2q_2)-v(k_1,k_2)=\alpha\left[v(\frac{k_1}{q_1},k_2)-v(k_1,k_2)\right]+\alpha\left[v(k_1q_1,k_2)-v(k_1,k_2)\right]$$

$$v(k_1,k_2)=\frac{1}{1-2\alpha}v(k_1,k_2q_2)-\frac{\alpha}{1-2\alpha}v(\frac{k_1}{q_1},k_2)-\frac{\alpha}{1-2\alpha}v(k_1q_1,k_2). \tag{2.36}$$

Replacing k_2 by k_2q_2 in (2.36) and continuing the same process, we arrive (2.30).

(iii) By replacing k_1 by $\dfrac{k_1}{q_1}$ in (2.36) repeatedly, we arrive (2.31).

(iv) By replacing k_1 by k_1q_1 in (2.36) repeatedly, we arrive (2.32). □

Corollary 2.6.2. *Let k_1, k_2 are non zero and $q_2\neq 1,0$. Then, we have*

$$\left\{\frac{k_1+k_2}{q_2-1}+\frac{log(k_1k_2)}{logq_2}\right\}-\left\{\frac{k_1+\frac{k_2}{q_2^m}}{q_2-1}+\frac{log\frac{(k_1k_2)}{q_2^m}}{logq_2}\right\}=\sum_{r=1}^{m}\left\{(k_2q_2^{-r})+1\right\}. \tag{2.37}$$

Proof. Taking $v(k_1,k_2)=\dfrac{k_1+k_2}{q_2-1}$, $u(k_1,k_2)=k_2$, $w(k_1,k_2)=1$ in (2.35), we get the proof of (2.37). □

Corollary 2.6.3. *Let k_1, k_2 are non zero, $\alpha \neq \frac{1}{2}$ and $q_1 \neq 0$. Then, we have*

$$k_1k_2 = \frac{1}{(1-2\alpha)^2}(k_1k_2q_2^2) - \frac{\alpha}{(1-2\alpha)^2}(\frac{k_1}{q_1}k_2q_2)$$

$$- \frac{\alpha}{(1-2\alpha)^2}(k_1q_1k_2q_2) - \frac{\alpha}{(1-2\alpha)}(\frac{k_1}{q_1}k_2) - \frac{\alpha}{(1-2\alpha)}(k_1q_1k_2). \tag{2.38}$$

Proof. The proof of (2.38) follows by taking $v(k_1, k_2) = k_1k_2$ and $m = 2$ in (2.30). □

Example 2.6.4. *Taking $k_1 = 0.3$, $k_2 = 0.2$, $q_2 = 2$ and $m = 2$ in (2.37), the value of both sides of (2.37) are equal and the value is 2.15.*

Formula (2.38) is verified by taking $k_1 = 4$, $k_2 = 2$, $q_2 = 2$, $q_1 = 3$, in this case value of both sides are equal, the common value is 8.

Corollary 2.6.5. *Let k_1, k_2 are non zero, $\alpha \neq \frac{1}{2}$ and $q_1 \neq 0$. Then, we have*

$$k_1k_2 = \frac{1}{1-2\alpha}(k_1k_2q_2) - \frac{\alpha}{(1-2\alpha)^2}(\frac{k_1}{q_1}k_2q_2) + \frac{\alpha^2}{(1-2\alpha)^2}(\frac{k_1}{q_1}^2k_2)$$

$$+ \frac{\alpha^2}{(1-2\alpha)^2}(k_1k_2) - \frac{\alpha}{1-2\alpha}(k_1q_1k_2). \tag{2.39}$$

Proof. Taking $v(k_1, k_2) = k_1k_2$ and $m = 1$ in (2.31), we get the proof of (2.39). □

Corollary 2.6.6. *Let k_1, k_2 are non zero, $\alpha \neq \frac{1}{2}$ and $q_1 \neq 0$. Then, we have*

$$k_1k_2 = \frac{1}{1-2\alpha}(k_1k_2q_2) - \frac{\alpha}{(1-2\alpha)}(\frac{k_1}{q_1}k_2) - \frac{\alpha}{(1-2\alpha)^2}(k_1q_1k_2q_2)$$

$$+ \frac{\alpha^2}{(1-2\alpha)^2}(k_1k_2) + \frac{\alpha^2}{1-2\alpha^2}(k_1q_1^2k_2). \tag{2.40}$$

Proof. The proof follows by taking $v(k_1, k_2) = k_1k_2$ and $m = 1$ in (2.32). □

Example 2.6.7. *Formula (2.39) and (2.40) are verified by taking $k_1 = 4$, $k_2 = 2$, $q_2 = 2$, $q_1 = 3$ and $k_1 = 4$, $k_2 = 2$, $q_1 = 3$ $q_2 = 2$, respectively.*

2.7 Two side flow q-heat equation with variable coefficient

In this section, we derive solutions of one dimensional two side q-heat equation with variable coefficient. Solutions are verified by MATLAB.

Theorem 2.7.1. *Four types of solutions of 1-D two side q-heat equation with variable coefficient are obtained as*

$$(i)\ v(k_1,k_2)-v\Big(k_1,\frac{k_2}{q_2^m}\Big)=\alpha(k_1,k_2)\sum_{r=1}^{m}\Big[v\Big(\frac{k_1}{q_1},k_2q_2^{-r}\Big)+v(k_1q_1,k_2q_2^{-r})-2v(k_1,k_2)\Big], \tag{2.41}$$

$$(ii)\ v(k_1,k_2)=\frac{1}{\prod\limits_{r=1}^{m}[1-2\alpha(k_1,k_2q_2^{r-1})]}v(k_1,k_2q_2^m)$$
$$-\sum_{r=1}^{m}\frac{\alpha(k_1,k_2q_2^{r-1})}{\prod\limits_{s=1}^{r}[1-2\alpha(k_1,k_2q_2^{s-1})]}\Big(v\Big(\frac{k_1}{q_1},k_2q_2^{r-1}\Big)+v(k_1q_1,k_2q_2^{r-1})\Big), \tag{2.42}$$

$$(iii)\ v(k_1,k_2)=\frac{1}{1-2\alpha(k_1,k_2)}v(k_1,k_2q_2)-\frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}v(k_1q_1,k_2)$$
$$-\frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)]\prod\limits_{r=1}^{m}[1-2\alpha(\frac{k_1}{q_1},k_2q_2^{r-1})]}v(\frac{k_1}{q_1},k_2q_2^m)$$
$$+\sum_{r=1}^{m}\frac{[\alpha(k_1,k_2)][\alpha(\frac{k_1}{q_1},k_2q_2^{r-1})]}{[1-2\alpha(k_1,k_2)]\prod\limits_{s=1}^{r}[1-2\alpha(\frac{k_1}{q_1},k_2q_2^{s-1})]}\Big\{v(\frac{k_1}{q_1^2},k_2q_2^{(r-1)})+v(k_1,k_2q_2^{(r-1)})\Big\}, \tag{2.43}$$

and $(iv)\ v(k_1,k_2)=\dfrac{1}{1-2\alpha(k_1,k_2)}v(k_1,k_2q_2)-\dfrac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}v(\dfrac{k_1}{q_1},k_2)$

$$-\frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)]\prod\limits_{r=1}^{m}[1-2\alpha(k_1q_1,k_2q_2^{r-1})]}v(k_1q_1,k_2q_2^m)$$

$$+\sum_{r=1}^{m}\frac{\alpha(k_1,k_2)\alpha(k_1q_1,k_2q_2^{r-1})\left\{v(k_1,k_2q_2^{(s-1)})+v(k_1q_1^2,k_2q_2^{(r-1)})\right\}}{[1-2\alpha(k_1,k_2)]\prod_{s=1}^{r}[1-2\alpha(k_1q_1,k_2q_2^{r-1})]}. \tag{2.44}$$

Proof. (i) From the linearity of $\underset{q_2;q_1}{\overset{-1}{\Delta}}$, (2.28), and assuming

$$v(k_1,k_2)-v(k_1,\frac{k_2}{q_2^m})=\alpha(k_1,k_2)\sum_{r=1}^{m}\left\{u(k_1,k_2q_2^{-r})+w(k_1,k_2q_2^{-r})\right\}, \tag{2.45}$$

$$\underset{q_1^{-1}}{\Delta}v(k_1,k_2)=u(k_1,k_2) \text{ and } \underset{q_1}{\Delta}v(k_1,k_2)=w(k_1,k_2). \tag{2.46}$$

Equation (2.46) is the solution of heat equation (2.28).

In (2.45), $u(k_1,k_2q_2)$ is obtained by replacing k_2 by k_2q_2 in (2.46) .

Substituting (2.46) in (2.45), we get (2.41).

(ii) From the q-Heat equation (2.28), we have

$$v(k_1,k_2)=\frac{v(k_1,k_2q_2)}{1-2\alpha(k_1,k_2)}-\frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}v(\frac{k_1}{q_1},k_2)-\frac{\alpha(k_1,k_2)v(k_1q_1,k_2)}{1-2\alpha(k_1,k_2)} \tag{2.47}$$

Replacing k_2 by k_2q_2 in (2.47), repeating the process we arrive (2.42).

(iii) Replacing k_1 by $\frac{k_1}{q_1}$ in (2.47), we arrive (2.43).

(iv) Replacing k_1 by k_1q_1 in (2.47), we arrive (2.44). □

Corollary 2.7.2. *Let $q_1 \neq 0$ and $\alpha(k_1,k_2),\alpha(k_1,k_2q_2)\neq\frac{1}{2}$, then we have*

$$k_1k_2=\frac{1}{[1-2\alpha(k_1,k_2)][1-2\alpha(k_1,k_2q_2)]}(k_1k_2q_2^2)-\frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)][1-2\alpha(k_1,k_2q_2)]}$$
$$\left\{\left(\frac{k_1}{q_1}k_2q_2\right)+(k_1q_1k_2q_2)\right\}-\frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)]}\left\{\left(\frac{k_1}{q_1}k_2\right)+(k_1q_1k_2)\right\}. \tag{2.48}$$

Proof. Taking $v(k_1,k_2)=k_1k_2$ and $m=2$ in (2.42), we get (2.48). □

Corollary 2.7.3. *Let $q_1 \neq 0$ and $\alpha(k_1,k_2) \neq \frac{1}{2}$, then we have*

$$k_1k_2 = \frac{1}{1-2\alpha(k_1,k_2)}(k_1k_2q_2) - \frac{\alpha(k_1,k_2)}{[1-2\alpha(k_1,k_2)][1-2\alpha\left(\frac{k_1}{q_1},k_2\right)]}(\frac{k_1}{q_1}k_2q_2)$$
$$+\frac{[\alpha(k_1,k_2)][\alpha\left(\frac{k_1}{q_1},k_2\right)]\left\{\left(\frac{k_1}{q_1}^2 k_2\right)+(k_1k_2)\right\}}{[1-2\alpha(k_1,k_2)][1-2\alpha\left(\frac{k_1}{q_1},k_2\right)]} - \frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}(k_1q_1k_2). \quad (2.49)$$

Proof. Taking $v(k_1,k_2) = k_1k_2$ and $m = 1$ in (2.43), we get the proof of (2.49). □

Corollary 2.7.4. *Let $q_1 \neq 0$ and $\alpha(k_1,k_2) \neq \frac{1}{2}$, then we have*

$$k_2k_1 = \frac{1}{1-2\alpha(k_1,k_2)}(k_1k_2q_2) - \frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}\left(\frac{k_1}{q_1}k_2\right) - \frac{\alpha(k_1,k_2)}{1-2\alpha(k_1,k_2)}(k_1q_1k_2q_2)$$
$$+\frac{\alpha(k_1,k_2)\alpha(k_1q_1,k_2)}{[1-2\alpha(k_1,k_2)][1-2\alpha(k_1q_1,k_2)]}\left\{k_2k_1+k_2k_1q_1^2\right\}. \quad (2.50)$$

Proof. Taking $v(k_1,k_2) = k_2k_1$ and $m = 1$ in (2.44), we get (2.50). □

From our results, it is possible to present heat flows in a rod by knowing the temperature at few positions. The results derived in this section can also be extended to the heat equation for both thin plate and medium.

Chapter 3

Discrete Delay Heat Equation Model

3.1 Introduction

In this chapter, we deal with the formulation of delay heat equation model. Here partial difference equation which extends its applications in heat equation is taken for study by the application of α-β difference operator and a model for heat transfer in the rod is found having recourse to Fourier law of cooling. An involved study is carried out to evaluate the movement of heat and thus numerous results are postulated. The results obtained are validated by MATLAB.

3.2 Discrete delay heat equation of a long rod

Consider a very long rod and the notations used in section 1.2. Even though we have assumed that heat flow is instantaneous, in reality, it takes time for heat to flow from one point k_1 to its neighbouring points $k_1 - \ell_1$ and $k_1 + \ell_1$ in one dimensional flow. Let γ be the positive diffusion rate constant of rod. By denoting $\underset{\alpha(\pm\ell_1,0)}{\Delta} = \underset{\alpha(\ell_1,0)}{\Delta} + \underset{\alpha(-\ell_1,0)}{\Delta}$, taking $n = 2$ in (1.3) and cooling law of Fourier, the discrete delay heat equation of rod is

$$\underset{\beta(0,\ell_2)}{\Delta} v(k_1,k_2) = \gamma \underset{\alpha(\pm\ell_1,0)}{\Delta} v(k_1,k_2-\sigma), \tag{3.1}$$

where σ is a delay factor. Here, we discuss the numerical solution of the discrete delay heat equation (3.1).

Theorem 3.2.1. *Assume that $m > 0$ is an integer, and $\ell_2 > 0$ which is real such that $v(k_1, k_2 - m\ell_2)$ and $\underset{\alpha(\pm\ell_1)}{\Delta} v(k_1,k_2-\sigma) = \underset{\alpha(\pm\ell_1)}{u}(k_1,k_2-\sigma)$ are well-known. Then the delay heat equation (3.1) has a solution $v(k_1,k_2)$ satisfying the relation*

$$v(k_1,k_2) = \beta^m v(k_1,k_2-m\ell_2) + \gamma \sum_{r=1}^{m} \beta^{r-1} \underset{\alpha(\pm\ell_1)}{u}(k_1,k_2-\sigma-r\ell_2). \tag{3.2}$$

Proof. By representing $\underset{\alpha(\pm\ell_1,0)}{\Delta} v(k_1,k_2) = \underset{\alpha(\pm\ell_1)}{u}(k_1,k_2-\sigma)$, from (1.3) and (3.1), we arrive

$$v(k_1,k_2) - \beta^m v(k_1,k_2-m\ell_2) = \gamma \underset{\beta(0,\ell_2)}{\overset{-1}{\Delta}} \underset{\alpha(\pm\ell_1)}{u}(k_1,k_2-\sigma)\Big|_{k-m\ell}^{k} \tag{3.3}$$

which yields (3.2). $\square$

Theorem 3.2.2. *Let $v(k_1, k_2)$ be a solution of (3.1) and denote $v(k_1 \pm \ell_1, *) = v(k_1 + \ell_1, *) + v(k_1 - \ell_1, *)$ and $v(*, k_2 \pm \ell_2) = v(*, k_2 + \ell_2) + v(*, k_2 - \ell_2)$. Then, we have the four identities*

$$
\begin{aligned}
\text{(a).} \quad v(k_1, k_2) &= \frac{1}{\beta^m} v(k_1, k_2 + m\ell_2) \\
&- \sum_{i=1}^{m} \frac{\gamma}{\beta^i} \big[v(k_1 \pm \ell_1, k_2 - \sigma + (i-1)\ell_2) - 2\alpha v(k_1, k_2 + (i-1)\ell_2 - \sigma) \big],
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
\text{(b).} \quad v(k_1, k_2) &= \beta^m v(k_1, k_2 - m\ell_2) + \sum_{i=1}^{m} \beta^{i-1} \gamma \big[v(k_1 + \ell_1, k_2 - i\ell_2 - \sigma) \\
&+ v(k_1 - \ell_1, k_2 - i\ell_2 - \sigma) - 2\alpha v(k_1, k_2 - i\ell_2 - \sigma) \big],
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
\text{(c).} \quad v(k_1, k_2) &= \frac{1}{\gamma^m} v(k_1 - m\ell_1, k_2 + m\ell_2 + m\sigma) - \sum_{i=1}^{m} \frac{\beta}{\gamma^i} [v(k_1 - i\ell_1, k_2 + (i-1)\ell_2 + i\sigma)] \\
&- \sum_{i=1}^{m} \frac{1}{\gamma^{i-1}} [v(k_1 - (i+1)\ell_1, k_2 + (i-1)\ell_2 + (i-1)\sigma)] \\
&+ \sum_{i=1}^{m} \frac{2\alpha}{\gamma^{i-1}} [v(k_1 - i\ell_1, k_2 + (i-1)\ell_2 + (i-1)\sigma)],
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\text{(d).} \quad v(k_1, k_2) &= \frac{1}{\gamma^m} v(k_1 + m\ell_1, k_2 + m\ell_2 + m\sigma) - \sum_{i=1}^{m} \frac{\beta}{\gamma^i} [v(k_1 + i\ell_1, k_2 + (i-1)\ell_2 + i\sigma)] \\
&+ \sum_{i=1}^{m} \frac{2\alpha}{\gamma^{i-1}} [v(k_1 + i\ell_1, k_2 + (i-1)\ell_2 + (i-1)\sigma)] \\
&- \sum_{i=1}^{m} \frac{1}{\gamma^{i-1}} [v(k_1 + (i+1)\ell_1, k_2 + (i-1)\ell_2 + (i-1)\sigma)].
\end{aligned} \tag{3.7}
$$

Proof. (a). From (3.1), we find

$$
\begin{aligned}
v(k_1, k_2) &= \frac{1}{\beta} v(k_1, k_2 + \ell_2) - \frac{\gamma}{\beta} \Big[v(k_1 + \ell_1, (k_2 - \sigma)) \\
&+ v(k_1 - \ell_1, k_2 - \sigma) - 2\alpha v(k_1, k_2 + \ell_2 - \sigma) \Big].
\end{aligned} \tag{3.8}
$$

By replacing k_2 by $k_2+\ell_2, k_2+2\ell_2, ..., k_2+m\ell_2$ in (3.8), we obtain expressions for $v(k_1,k_2+i\ell_2)$ and $v(k_1\pm\ell_1,k_2+i\ell_2)$. Now proof of (a) follows by applying all these values in (3.8).

(b). The delay heat equation (3.1) generates

$$v(k_1,k_2)=\beta v(k_1,k_2-\ell_2)+\gamma\Big[v(k_1+\ell_1,k_2-\ell_2-\sigma)$$
$$+v(k_1-\ell_1,k_2-\ell_2-\sigma)-2\alpha v(k_1,k_2-\ell_2-\sigma)\Big]. \tag{3.9}$$

Proof follows by replacing k_2 by $k_2-\ell_2, k_2-2\ell_2, ..., k_2-m\ell_2$ repeatedly and substituting corresponding γ-values in (3.9).

(c). A simple calculation on (3.1) gives the expression

$$v(k_1,k_2)=\frac{1}{\gamma}v(k_1-\ell_1,k_2+\ell_2+\sigma)-\frac{\beta}{\gamma}v(k_1-\ell_1,k_2+\sigma)-v(k_1-2\ell_1,k_2)+2\alpha v(k_1-\ell_1,k_2).$$

By replacing k_1 by $k_1-\ell_1, k_1-2\ell_1, ..., k_1-m\ell_1$ and k_2 by $k_2+\ell_2+\sigma, k_2+2\ell_2+\sigma, ..., k_2+m\ell_2+\sigma$ repeatedly and applying these values complete the proof.

(d). From (3.1), $v(k_1,k_2)=\frac{1}{\gamma}v(k_1+\ell_1,k_2+\ell_2+\sigma)-\frac{\beta}{\gamma}v(k_1+\ell_1,k_2+\sigma)$
$$-v(k_1+2\ell_1,k_2)+2\alpha v(k_1+\ell_1,k_2),$$

Replacing k_1 by $k_1+\ell_1, k_1+2\ell_1, ..., k_1+m\ell_1$ and k_2 by $k_2+\ell_2+\sigma, k_2+2\ell_2+\sigma, ...,$ $k_2+m\ell_2+\sigma$, we obtain (3.7). □

Example 3.2.3. *The dissemination rate of rod is identified by the given example if the solution $v(k_1,k_2)$ of (3.1) is known. If we try $v(k_1,k_2)=e^{k_1+k_2}$ as a closed form solution of (3.1), then we have* $\underset{\beta(0,\ell_2)}{\Delta} e^{k_1+k_2}=\gamma\big[\underset{\alpha(\ell_1,0)}{\Delta} e^{k_1+k_2-\sigma}+\underset{\alpha(-\ell_1,0)}{\Delta} e^{k_1+k_2-\sigma}\big]$, *which yields*

$e^{k_1+k_2+\ell_2} - \beta e^{k_1+k_2} = \gamma\big[e^{k_1+k_2+\ell_1-\sigma} + e^{k_1+k_2-\ell_1-\sigma} - 2\alpha e^{k_1+k_2-\sigma}\big]$ *and*

$$\gamma = \frac{e^{\ell_2} - \beta}{e^{\ell_1-\sigma} + e^{-\ell_1-\sigma} - 2\alpha e^{-\sigma}}. \tag{3.10}$$

We use the following MATLAB coding to verify (a) of Theorem 3.2.2 for $m = 15$, $\sigma = k_1 = \ell_1 = 1$, $k_2 = \ell_2 = \alpha = 2$, $\beta = 3$, $v(k_1, k_2) = e^{(k_1+k_2)}$, *and* γ *is as in* (3.10).

$exp(3) = ((1./(3). \wedge 1). * (exp(5))) - symsum(((-13.05557647)./3. \wedge i). * (((exp(3 + (i-1). * 2))) + exp(1 + (i-1). * 2) - (4. * (exp(2 + (i-1). * 2)))), i, 1, 1).$

3.3 Discrete delay heat equation for thin plate

Assume that $v(k_1, k_2, k_3)$ be the temperature of thin plate at real position (k_1, k_2) and at time k_3. As in the case of rod, the partial β-α delay heat equation for the thin plate can be formulated as

$$\underset{\beta(0,0,\ell_3)}{\Delta} v(k_1, k_2, k_3) = \gamma \underset{\alpha(\pm\ell_{(1,2)})}{\Delta} v(k_1, k_2, k_3 - \sigma), \tag{3.11}$$

where $\underset{\alpha(\pm\ell_{(1,2)})}{\Delta} = \underset{\alpha(\ell_1,0,0)}{\Delta} + \underset{\alpha(-\ell_1,0,0)}{\Delta} + \underset{\alpha(0,\ell_2,0)}{\Delta} + \underset{\alpha(0,-\ell_2,0)}{\Delta}$ and σ is a delay factor.

Theorem 3.3.1. *Let* $m > 0$ *be integer and* $\ell_3 > 0$ *such that* $v(k_1, k_2, k_3 - ml_3)$ *and partial differences* $\underset{\alpha(\pm\ell_{(1,2)})}{\Delta} v(k_1, k_2, k_3) = \underset{\alpha(\pm\ell_{(1,2)})}{u}(k_1, k_2, k_3 - \sigma)$ *are known functions. Then, a solution* $v(k_1, k_2, k_3)$ *of* (3.11) *satisfies the relation*

$$v(k_1, k_2, k_3) = \beta^m v(k_1, k_2, k_3 - m\ell_3) + \gamma \sum_{r=1}^{m} \beta^{r-1} \underset{\alpha(\pm\ell_{(1,2)})}{u}(k_1, k_2, k_3 - \sigma - r\ell_3). \tag{3.12}$$

Proof. The proof of (3.12) is similar to Theorem 3.2.2. $\square$

Consider the following notations which will be used in the subsequent theorems:

$$v(k_{(1,2)} \pm l_{(1,2)}, *) = v(k_1 \pm \ell_1, k_2, *) + v(k_1, k_2 \pm \ell_2, *),$$

$$v(k_{(2,3)} \pm l_{(2,3)}, *) = v(*, k_2 \pm \ell_2, k_3) + v(*, k_2, k_3 \pm \ell_3),$$

$$v(k_1 \pm \ell_1, *, *) = v(k_1 + \ell_1, *, *) + v(k_1 - \ell_1, *, *),$$

$$v(*, *, k_3 \pm \ell_3) = v(*, *, k_3 - \sigma + \ell_3) + v(*, *, k_3 - \sigma - \ell_3).$$

Theorem 3.3.2. *Assume that $v(k_1, k_2, k_3)$ is a solution of equation (3.11), and $v(k_1 \pm r\ell_1, k_2 \pm r\ell_2)$ exist. Then, we obtain four types of solutions of (3.11) as*

(a). $$v(k_1, k_2, k_3) = \frac{1}{\beta^m} v(k_1, k_2, k_3 + m\ell_3) - \sum_{i=1}^{m} \frac{\gamma}{\beta^i} \Big[v(k_1 \pm \ell_1, k_2, k_3 + (i-1)\ell_3 - \sigma) + v(k_1, k_2 \pm \ell_2, k_3 + (i-1)\ell_3 - \sigma) \Big] - 4\alpha v(k_1, k_2, k_3 + (i-1)\ell_3 - \sigma) \big], \quad (3.13)$$

(b). $$v(k_1, k_2, k_3) = \beta^m v(k_1, k_2, k_3 - m\ell_3) + \sum_{i=1}^{m} \beta^{(i-1)} \gamma \big[v(k_1 \pm \ell_1, k_2, k_3 - \sigma - i\ell_3) + v(k_1, k_2 \pm \ell_2, k_3 - \sigma - i\ell_3) - 4\alpha v(k_1, k_2, k_3 - \sigma - i\ell_3) \big], \quad (3.14)$$

(c). $$\begin{aligned} v(k) = {} & \frac{1}{\gamma^m} v(k_1 - i\ell_1, k_2, k_3 + i\ell_3 + i\sigma) - \sum_{i=1}^{m} \frac{\beta}{\gamma^i} v(k_1 - i\ell_1, k_2, k_3 + (i-1)\ell_3 + i\sigma) \\ & - \sum_{i=1}^{m} \frac{1}{\gamma^{i-1}} v(k_1 - (i+1)\ell_1, k_2, k_3 + (i-1)\ell_3 + (i-1)\sigma) \\ & - \sum_{i=1}^{m} \frac{1}{\gamma^{i-1}} v(k_1 - i\ell_1, k_2 \pm \ell_2, k_3 + (i-1)\ell_3 + (i-1)\sigma) \\ & - \sum_{i=0}^{m} \frac{4\alpha}{\gamma^{i-1}} v(k_1 - i\ell_1, k_2, k_3 + (i-1)\ell_3 + (i-1)\sigma), \end{aligned} \quad (3.15)$$

(d). $$\begin{aligned} v(k) = {} & \frac{1}{\gamma^m} v(k_1 + i\ell_1, k_2, k_3 + i\sigma + i\ell_3) - \sum_{i=1}^{m} \frac{\beta}{\gamma^i} v(k_1 + i\ell_1, k_2, k_3 + (i-1)\ell_3 + i\sigma) \\ & - \sum_{i=1}^{m} v(k_1 + (i+1)\ell_1, k_2, k_3 + (i-1)\ell_3 + (i-1)\sigma) \end{aligned}$$

$$- \sum_{i=1}^{m} \frac{1}{\gamma^{i-1}} v(k_1 + i\ell_1, k_2 \pm \ell_2, k_3 + (i-1)\ell_3 + (i-1)\sigma)$$

$$+ \sum_{i-1}^{m} \frac{4\alpha}{\gamma^{i-1}} v(k_1 + i\ell_1, k_2, k_3 + (i-1)\ell_3 + (i-1)\sigma). \tag{3.16}$$

Proof. The proof is similar as Theorem 3.2.2. □

In most general case, consider homogeneous diffusion medium in $\Re^3$. Let $v(k_1, k_2, k_3, k_4, k_5)$ be the temperature, at position (k_1, k_2, k_3), at time k_4 with density (or pressure) k_5 and denote $k = (k_1, k_2, k_3, k_4, k_5)$. The partial β-α delay heat equation for a homogenous medium is expressed as

$$\underset{\beta(\ell_4,\ell_5)}{\Delta} v(k) = \gamma \underset{\alpha(\pm\ell_{(1,2,3)})}{\Delta} v(k-\sigma), \tag{3.17}$$

where $\underset{\alpha(\pm\ell_{(1,2,3)})}{\Delta} = \underset{\alpha(\ell_1)}{\Delta} + \underset{\alpha(-\ell_1)}{\Delta} + \underset{\alpha(\ell_2)}{\Delta} + \underset{\alpha(-\ell_2)}{\Delta} + \underset{\alpha(\ell_3)}{\Delta} + \underset{\alpha(-\ell_3)}{\Delta}$ and σ is a delay factor. As in the previous cases, equation (3.17) has four types of solutions as given below

(a). $v(k) = \frac{1}{\beta^m} v(k - m\ell_{4,5}) - \sum_{i=1}^{m} \frac{\gamma}{\beta^i} \Big[v(k + (\pm\ell_1, 0, 0, (i-1)\ell_4 - \sigma, (i-1)\ell_5 - \sigma)$

$$+ v(k + (0, \pm\ell_2, 0, (i-1)\ell_4 - \sigma, (i-1)\ell_5 - \sigma))$$

$$+ v(k + (0, 0, \pm\ell_3, (i-1)\ell_4 - \sigma, (i-1)\ell_5 - \sigma))$$

$$- 6\alpha v(k + (0, 0, 0, (i-1)\ell_4 - \sigma), (i-1)\ell_5 - \sigma) \Big], \tag{3.18}$$

(b). $v(k) = \beta^m v(k - m\ell_{4,5})$

$$+ \sum_{i=1}^{m} \beta^{i-1} \gamma \Big[v(k + (\pm\ell_1, 0, 0, -i\ell_4 - \sigma, -i\ell_5 - \sigma))$$

$$+v(k+(0,\pm\ell_2,0,-i\ell_4-\sigma,-i\ell_5-\sigma))+v(k+(0,0,\pm\ell_3,-i\ell_4-\sigma,-i\ell_5-\sigma))$$

$$-6\alpha v(k+(0,0,0,-\sigma-i\ell_4,-\sigma-i\ell_5))\Big], \tag{3.19}$$

$$\begin{aligned}
\text{(c). } v(k) = {} & \frac{1}{\gamma^m}v(k+(-m\ell_1,0,0,m\ell_4+m\sigma,m\ell_5+m\sigma)) \\
& -\sum_{i=1}^{m}\frac{\beta}{\gamma^i}v(k+(-i\ell_1,0,0,(i-1)\ell_4+i\sigma,(i-1)\ell_5+i\sigma)) \\
& -\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v(k+(-(i+1)\ell_1,0,0,(i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1)\sigma)) \\
& -\sum_{i=0}^{m}\frac{1}{\gamma^{i-1}}v(k+(-i\ell_1,\pm\ell_2,0,(i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1)\sigma)) \\
& -\sum_{i-1}^{m}\frac{1}{\gamma^{i-1}}v(k+(-i\ell_1,0,\pm\ell_3,(i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1\sigma)))
\end{aligned}$$

$$-\sum_{i=0}^{m}\frac{6\alpha}{\gamma^{i-1}}v(k+(-i\ell_1,0,0,(i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5)+(i-1)\sigma)), \tag{3.20}$$

$$\begin{aligned}
\text{(d). } v(k) = {} & \frac{1}{\gamma^m}v(k+(\ell_1,0,0,m\ell_4+m\sigma,m\ell_5+m\sigma)) \\
& -\sum_{i=1}^{m}\frac{\beta}{\gamma^i}v(k+(i\ell_1,0,0,(i-1)\ell_4+i\sigma,(i-1)\ell_5+i\sigma)) \\
& -\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v(k+((i+1)\ell_1,0,0,(i+1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1)\sigma)) \\
& -\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v(k+(i\ell_1,\pm\ell_2,0,(i-1)\ell_4+(i-1)\sigma,(i-1)\ell_5+(i-1)\sigma)) \\
& -\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v(k+(i\ell_1,0,\pm\ell_3,(i-1)\sigma+(i-1)\ell_5,(i-1)\sigma))
\end{aligned}$$

$$-\sum_{i=1}^{m}\frac{6\alpha}{\gamma^{i-1}}v(k+(i\ell_1,0,0,(i-1)\ell_4+(i-1)\sigma,(i-1)\ell_4+(i-1)\sigma)). \tag{3.21}$$

3.4 Discrete delay q-heat equation of a long rod

Consider a long rod and assuming $v(k_1,k_2)$ be the temperature at the position k_1 and its time k_2 of a rod. Let γ be the positive dissemination rate constant of rod.

By Fourier's cooling law, discrete α-β delay q-heat equation of rod is

$$v(k_1,k_2q_2)-\beta v(k_1,k_2)=\gamma\Big[v\big(\frac{k_1}{q_1},\frac{k_2}{\sigma}\big)-\alpha v\big(k_1,\frac{k_2}{\sigma}\big)+v\big(k_1q_1,\frac{k_2}{\sigma}\big)-\alpha v\big(k_1,\frac{k_2}{\sigma}\big)\Big]. \quad (3.22)$$

Then the discrete heat equation (3.22) with delay can be expressed as

$$\underset{\beta(1,q_2{}^{\pm})}{\Delta} v(k_1,k_2)=\gamma \underset{\alpha(q_1{}^{\pm},1)}{\Delta} v\big(k_1,\frac{k_2}{\sigma}\big). \quad (3.23)$$

Here, we analyze the possible solutions of the q-heat equation (3.23).

Theorem 3.4.1. *Let $m>0$ be integer, and $q_2>0$ is real such that $v(k_1,\frac{k_2}{q_2^m})$ and $\underset{\alpha(q_1^{\pm},1)}{\Delta} v(k_1,\frac{k_2}{\sigma})=\underset{\alpha(q_1^{\pm},1)}{u}(k_1,\frac{k_2}{\sigma})$ are given. Then (3.23) has a summation solution as*

$$v(k_1,k_2)=\alpha^m v\big(k_1,\frac{k_2}{q_2^m}\big)+\gamma\sum_{r=1}^{m}\alpha^r \underset{\alpha(q_1^{\pm},1)}{u}\big(k_1,\frac{k_2}{q_2^r\sigma}\big). \quad (3.24)$$

Proof. Taking $\underset{\alpha(q_1^{\pm},1)}{\Delta} v(k_1,\frac{k_2}{\sigma})=\underset{\alpha(q_1^{\pm},1)}{u}(k_1,\frac{k_2}{\sigma})$ in (3.23) gives

$$v(k_1,k_2)=\gamma \underset{\beta(1,q_2)}{\Delta^{-1}} \underset{\alpha(q_1^{\pm},1)}{u}(k_1,\frac{k_2}{\sigma}). \quad (3.25)$$

The proof follows by applying inverse principle on (3.25). □

Theorem 3.4.2. *When $\beta>0$, m is a positive integer and by denoting $v(k_1q_1^{\pm},*)=v(k_1q_1,*)+v(\frac{k_1}{q_1},*)$ and $v(*,k_2q_2{}^{\pm})=v(*,k_2q_2)+v(*,\frac{k_2}{q_1})$, we get the following:*

$$(a).\ v(k_1,k_2)=\frac{1}{\beta^m}v(k_1,k_2q_2{}^m)-\sum_{i=1}^{m}\frac{\gamma}{\beta^i}\Big[v\big(k_1q_1^{\pm},\frac{k_2q_2^{i-1}}{\sigma}\big)-2\alpha v\big(k_1,\frac{k_2q_2^{i-1}}{\sigma}\big)\Big], \quad (3.26)$$

$$(b).\ v(k_1,k_2)=\beta^m v\big(k_1,\frac{k_2}{q_2^m}\big)+\sum_{i=1}^{m}\beta^{i-1}\gamma\Big[v\big(k_1q_1^{\pm},\frac{k_2}{q_2^i\sigma}\big)-2\alpha v\big(k_1,\frac{k_2}{q_2^i\sigma}\big)\Big], \quad (3.27)$$

(c). $v(k_1,k_2)=\frac{1}{\gamma^m}v\big(\frac{k_1}{q_1^m},k_2q_2^m\sigma^m\big)-\sum_{i=1}^{m}\frac{\beta}{\gamma^i}v\big(\frac{k_1}{q_1^i},k_2q_2^{i-1}\sigma^i\big)$

$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v\big(\frac{k_1}{q_1^{i+1}},k_2q_2^{i-1}\sigma^{i-1}\big)-\sum_{i=1}^{m}\frac{2\alpha}{\gamma^{i-1}}v\big(\frac{k_1}{q_1^i},k_2q_2^{i-1}\sigma^{i-1}\big), \tag{3.28}$$

(d). $v(k_1,k_2)=\frac{1}{\gamma^m}v(k_1q_1^m,k_2q_2^m\sigma^m)-\sum_{i=1}^{m}\frac{\beta}{\gamma^i}v\big(k_1q_1^i,k_2q_2^{i-1}\sigma^i\big)$

$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}v\big(k_1q_1^{i+1},k_2q_2^{i-1}\sigma^{i-1}\big)-\sum_{i=1}^{m}\frac{2\alpha}{\gamma^{i-1}}v\big(k_1q_1^i,k_2q_2^{i-1}\sigma^{i-1}\big). \tag{3.29}$$

Proof. (a). From (3.23), we have

$$v(k_1,k_2)=\frac{1}{\beta}v(k_1,k_2q_2)-\frac{\gamma}{\beta}\Big[v\big(k_1q_1^{\pm},\frac{k_2}{\sigma}\big)-2\alpha v\big(k_1,\frac{k_2}{\sigma}\big)\Big]. \tag{3.30}$$

Replacing k_2 by $k_2q_2,k_2q_2^2,.....,k_2q_2^m$ in (3.30), we get the result (3.26).

(b). From (3.23), we get

$$v(k_1,k_2)=\beta v\big(k_1,\frac{k_2}{q_2}\big)-\gamma\Big[v\big(k_1q_1^{\pm},\frac{k_2}{q_2\sigma}\big)-2\alpha v\big(k_1,\frac{k_2}{q_2\sigma}\big)\Big]. \tag{3.31}$$

By changing k_2 by $\frac{k_2}{q_2},\frac{k_2}{q_2^2},....,\frac{k_2}{q_2^m}$ repeatedly, we get the result (3.27).

(c). (3.23) yields $v(k_1,k_2)=\frac{1}{\gamma}v\big(\frac{k_1}{q_1},k_2q_2\sigma\big)-\frac{\beta}{\gamma}v\big(\frac{k_1}{q_1},\sigma k_2\big)-v\big(\frac{k_1}{{q_1}^2},k_2\big)+2\alpha v\big(\frac{k_1}{q_1},k_2\big)$. By replacing k_1 by $\frac{k_1}{q_1},\frac{k_1}{q_1^2},...,\frac{k_1}{q_1^m}$ and k_2 by $k_2q_2\sigma,k_2q_2^2\sigma^2,....,k_2q_2^m\sigma^m$ repeatedly in the above relation, we obtain (3.28).

(d). The proof of (3.29) follows by replacing k_1 by $k_1q_1,k_1q_1^2,...,k_1q_1^m$ k_2 by $k_2q_2\sigma,k_2q_2^2\sigma^2,....,k_2q_2^m\sigma^m$ repeatedly in (3.23) which is expressed as

$v(k_1,k_2)=\frac{1}{\gamma}v(k_1q_1,k_2q_2\sigma)-\frac{\beta}{\gamma}v(k_1q_1,\sigma k_2)-v\big(k_1{q_1}^2,k_2\big)+2\alpha v\big(k_1q_1,k_2\big).$ □

Following example is an illustration for (a) of Theorem 3.4.2.

Example 3.4.3. *Suppose that* $v(k_1,k_2) = k_1k_2$ *is a exact solution of* (3.23), $v(k_1,k_2) = \gamma[\underset{(q_1,1)}{\Delta} k_1k_2 + \underset{(\frac{1}{q_1},1)}{\Delta} k_1k_2]$ *yields* $k_1k_2q_2 - \beta k_1k_2 = \gamma[k_1q_1k_2 + \frac{k_1}{q_1}k_2 - 2k_1k_2]$. *Cancelling* k_1k_2 *on the both sides derives* $\gamma = \dfrac{q_2-\beta}{\dfrac{q_1}{\sigma}+\dfrac{1}{q_1\sigma}-\dfrac{2\alpha}{\sigma}}$.

For numerical verification, we give the MATLAB coding for (a) by taking

$k_1 = q_1 = 4,\ k_2 = q_2 = 5,\ \beta = \sigma = 3,\ \alpha = 2,\ m = 20,$

$4.*5 = ((1./(3).\wedge 20).*(20.*(5.\wedge 20))) - symsum((24./(3.\wedge i)).*((16.*(5.*(5.\wedge (i-1)))./3) + (1.*(5.*(5.\wedge (i-1)))./3) - (4.*(20.*((5.\wedge (i-1))./3)))), i, 1, 20).$

3.5 Discrete delay q-heat equation for thin plate

Let $v(k_1,k_2,k_3)$ be the temperature of a thin plate at position (k_1,k_2) and time k_3. The proportional amount of heat flows from left to right at the position $k = (k_1,k_2,k_3)$ is $\underset{(\frac{1}{q_1},1,1)}{\Delta} v(k)$, right to left is $\underset{(q_1,1,1)}{\Delta} v(k)$, top to bottom is $\underset{(1,q_2,1)}{\Delta} v(k)$ and bottom to top is $\underset{(1,\frac{1}{q_2},1)}{\Delta} v(k)$. By Fourier law of cooling and denoting $\underset{(q_1q_2)^{\pm}}{\Delta} = \underset{(q_1,1,1)}{\Delta} + \underset{(\frac{1}{q_1},1,1)}{\Delta} + \underset{(1,q_2,1)}{\Delta} + \underset{(1,\frac{1}{q_2},1)}{\Delta}$ the heat equation for the plate is

$$\underset{\beta(1,1,q_3)}{\Delta} v(k_1,k_2,k_3) = \gamma \underset{\alpha(q_1,q_2)^{\pm}}{\Delta} v(k_1,k_2,\frac{k_3}{\sigma}). \tag{3.32}$$

Theorem 3.5.1. *Let* $m > 0$ *and* $q_3 > 0$ *such that* $v(k_1,k_2,\frac{k_3}{q_3^m})$ *and the partial differences* $\underset{\alpha(q_1^{\pm},1)}{\Delta} v(k_1,k_2,\frac{k_3}{\sigma}) = \underset{(q_1^{\pm},1)}{u}(k_1,k_2,\frac{k_3}{\sigma})$ *are known, then we have*

$$v(k) = \alpha^m v(k_1,k_2,\frac{k_3}{q_3^m}) + \gamma\sum_{r=1}^{m}\alpha^r \underset{\alpha(q_1^{\pm},1)}{u}(k_1,k_2,\frac{k_3}{\sigma q_3^r}). \tag{3.33}$$

Proof. Taking $\underset{\alpha(q_1^{\pm},1)}{\Delta} v(\frac{k}{\sigma}) = \underset{\alpha(q_1^{\pm},1)}{u} (\frac{k}{\sigma})$ in (3.32), we arrive at

$$v(k) = \gamma \underset{\beta(1,1,q_3)}{\Delta^{-1}} \underset{\alpha(q_1^{\pm},1)}{u} \Big(\frac{k}{\sigma}\Big). \tag{3.34}$$

By using the inverse principle of $\underset{\beta(q_3)}{\overset{-1}{\Delta}}$ on (3.34), we obtain (3.33). $\square$

Consider the following notation which will be used in Theorem 3.5.2

$v(k_{(1,2)}q_{(1,2)},*)^{\pm} = v(k_1{q_1}^{\pm},k_2,*) + v(k_1,k_2{q_2}^{\pm},*)$ also

$v(k_{(2,3)}q_{(2,3)},*)^{\pm} = v(*,k_2{q_2}^{\pm},k_3) + v(*,k_2,k_3{q_3}^{\pm})$.

Theorem 3.5.2. *Assuming (3.32), then we have the following identities*

(a). $$v(k_1,k_2,k_3) = \frac{1}{\beta^m} v\big(k_1,k_2,k_3{q_3}^m\big) - \sum_{i=1}^{m} \frac{\gamma}{\beta^i}\Big[v\Big(k_1q_1^{\pm},k_2,\frac{k_3q_3^{i-1}}{\sigma}\Big) + v\Big(k_1,k_2q_2^{\pm},\frac{k_3q_3^{i-1}}{\sigma}\Big) - 4\alpha v\Big(k_1,k_2,\frac{k_3q_3^{i-1}}{\sigma}\Big)\Big], \tag{3.35}$$

(b). $$v(k_1,k_2,k_3) = \beta^m v\Big(k_1,k_2,\frac{k_3}{q_3^m}\Big) + \sum_{i=1}^{m} \beta^{i-1}\gamma\Big[v\Big(k_1q_1^{\pm},k_2,\frac{k_3}{q_3^i\sigma}\Big) + v\Big(k_1,k_2q_2^{\pm},\frac{k_3}{q_3^i\sigma}\Big) - 4\alpha v\Big(k_1,k_2,\frac{k_3}{q_3^i\sigma}\Big)\Big], \tag{3.36}$$

(c). $$\begin{aligned} v(k_1,k_2,k_3) = {} & \frac{1}{\gamma^m} v\Big(\frac{k_1}{q_1^m},k_2,k_3q_3^m\sigma^m\Big) - \sum_{i=1}^{m} \frac{\beta}{\gamma^i} v\Big(\frac{k_1}{q_1^i},k_2,k_3q_3^{i-1}\sigma^i\Big) \\ & - \sum_{i=1}^{m} \frac{1}{\gamma^{i-1}}\Big[v\Big(\frac{k_1}{q_1^{i+1}},k_2,k_3q_3^{i-1}\sigma^{i-1}\Big) + v\Big(\frac{k_1}{q_1^i},k_2q_2^{\pm},k_3q_3^{i-1}\sigma^{i-1}\Big)\Big] \\ & - \sum_{i=1}^{m} \frac{4\alpha}{\gamma^{i-1}} v\Big(\frac{k_1}{q_1^i},k_2,k_3q_3^{i-1}\sigma^{i-1}\Big), \end{aligned} \tag{3.37}$$

(d). $$v(k_1,k_2,k_3) = \frac{1}{\gamma^m} v\big(k_1q_1^m,k_2,k_3q_3^m\sigma^m\big) - \sum_{i=1}^{m} \frac{\beta}{\gamma^i} v\big(k_1q_1^i,k_2,k_3q_3^{i-1}\sigma^i\big)$$

$$-\sum_{i=1}^{m}\frac{1}{\gamma^{i-1}}\Big[v\big(k_1q_1^{i+1},k_2,k_3q_3^{i-1}\sigma^{i-1}\big)+v\big(k_1q_1^{i},k_2q_2^{\pm},k_3q_3^{i-1}\sigma^{i-1}\big)\Big]$$

$$-\sum_{i=1}^{m}\frac{4\alpha}{\gamma^{i-1}}v\big(k_1q_1^{i},k_2,k_3q_3^{i-1}\sigma^{i-1}\big). \tag{3.38}$$

Proof. The proof and verification are as similar as in the case of long rod. □

3.6 Discrete delay q-heat equation of medium

By the Fourier law of cooling, the heat equation for medium in $\Re^3$ is

$$\underset{\beta(q_4,q_5)}{\Delta} v(k) = \gamma \underset{\alpha(q_1,q_2,q_3)^{\pm}}{\Delta} v\Big(\frac{k}{\sigma}\Big), \tag{3.39}$$

where $\underset{(q_1,q_2,q_3)^{\pm}}{\Delta} = \underset{(q_1)}{\Delta} + \underset{(\frac{1}{q_1})}{\Delta} + \underset{(q_2)}{\Delta} + \underset{(\frac{1}{q_2})}{\Delta} + \underset{(q_3)}{\Delta} + \underset{(\frac{1}{q_3})}{\Delta}$ and $k=(k_1,k_2,k_3,k_4,k_5)$.

Theorem 3.6.1. *Assume that the function* $\underset{\alpha(q_1\,q_2,q_3)^{\pm}}{\Delta} v(\frac{k}{\sigma}) = \underset{\alpha(q_1,q_2,q_3)^{\pm}}{u}(\frac{k}{\sigma})$ *is known. Then, the solution* $v(k)$ *of the heat equation* (3.39) *satisfies the relation*

$$v(k) = v\left(k_1,k_2,k_3,\frac{k_4}{{q_4}^m},\frac{k_5}{{q_5}^m}\right) + \gamma\sum_{r=1}^{m} \underset{\alpha(q_1,q_2,q_3)^{\pm}}{u}\left(k_1,k_2,k_3,\frac{k_4}{{q_4}^r\sigma},\frac{k_5}{{q_5}^r\sigma}\right). \tag{3.40}$$

Proof. Taking $\underset{\alpha(q_1,q_2,q_3)^{\pm}}{\Delta} v(\frac{k}{\sigma}) = \underset{\alpha(q_1,q_2,q_3)^{\pm}}{u}(\frac{k}{\sigma})$ in (3.39), we get

$$v(k) = \gamma \underset{\beta(q_4,q_5)}{\Delta^{-1}} \underset{\alpha(q_1,q_2,q_3)^{\pm}}{u}\left(\frac{k}{\sigma}\right). \tag{3.41}$$

Then using the inverse principle in (3.41), we get (3.40). □

In the below theorem, we use the following notations:

$$v(k_{(1,2,3)}q^{\pm}_{(1,2,3)},*,*) = v(k_1q_1,k_2,k_3,*,*) + v(\tfrac{k_1}{q_1},k_2,k_3,*,*)$$
$$+ v(k_1,k_2q_2,k_3,*,*) + v(k_1,\tfrac{k_2}{q_2},k_3,*,*)$$

$$+ v(k_1, k_2, k_3q_3, *, *) + v(k_1, k_2, \tfrac{k_3}{q_3}, *, *).$$

$$v(*, k_{(2,3)}q^{\pm}_{(2,3)}, *, *) = v(*, k_2q_2, k_3, *, *) + v(*, \tfrac{k_2}{q_2}, k_3, *, *)$$

$$+ v(*, k_2, k_3q_3, *, *) + v(*, k_2, \frac{k_3}{q_3}, *, *).$$

Theorem 3.6.2. *If $v(k)$ is a solution of the equation (3.39) and $m > 0$ and $k = (k_1, k_2, k_3, k_4, k_5)$, then the following relations are equivalent:*

(a). $v(k, k_4, k_5) = \frac{1}{\beta^m} v(k, k_4{q_4}^m, k_5{q_5}^m) - \sum\limits_{i=1}^{m} \dfrac{\gamma}{\beta^i}\Big[v\big(kq_1^{\pm}, \frac{k_4q_4^{i-1}}{\sigma}, \dfrac{k_5q_5^{i-1}}{\sigma}\big)$

$$+v\big(kq_2^{\pm}, \frac{k_4q_4^{i-1}}{\sigma}, \frac{k_5q_5^{i-1}}{\sigma}\big) + v\big(kq_3^{\pm}, \frac{k_4q_4^{i-1}}{\sigma}, \frac{k_5q_5^{i-1}}{\sigma}\big) - 6\alpha v\big(k, \frac{k_4q_4^{i-1}}{\sigma}, \frac{k_5q_5^{i-1}}{\sigma}\big)\Big], \tag{3.42}$$

(b). $v(k, k_4, k_5) = \beta^m v\big(k, \frac{k_4}{q_4^m}, \frac{k_5}{q_5^m}\big) + \sum\limits_{i=1}^{m} \beta^{i-1}\gamma\Big[v\big(kq_1^{\pm}, \frac{k_4}{q_4^i\sigma}, \dfrac{k_5}{q_5^i\sigma}\big)$

$$+ v\big(kq_2^{\pm}, \frac{k_4}{q_4^i\sigma}, \frac{k_5}{q_5^i\sigma}\big) + v\big(kq_3^{\pm}, \frac{k_4}{q_4^i\sigma}, \frac{k_5}{q_5^i\sigma}\big) - 6\alpha v\big(k, \frac{k_4}{q_4^i\sigma}, \frac{k_5}{q_5^i\sigma}\big)\Big], \tag{3.43}$$

(c). $v(k, k_4, k_5) = \frac{1}{\gamma^m} v\big(\frac{k}{q_1^m}, k_4q_4^m\sigma^m, k_5q_5^m\sigma^m\big) - \sum\limits_{i=1}^{m} \frac{\beta}{\gamma^i} v\big(\frac{k}{q_1^i}, k_4q_4^{i-1}\sigma^i, k_5q_5^{i-1}\sigma^i\big)$

$$- \sum_{i=1}^{m} \tfrac{1}{\gamma^{i-1}}\Big[v\big(\tfrac{k}{q_1^{i+1}}, k_4q_4^{i-1}\sigma^{i-1}, k_5q_5^{i-1}\sigma^{i-1}\big) + v\big(\tfrac{kq_2^{\pm}}{q_1^i}, k_4q_4^{i-1}\sigma^{i-1}, k_5q_5^{i-1}\sigma^{i-1}\big)\Big]$$

$$- \sum_{i=1}^{m} \frac{6\alpha}{\gamma^{i-1}} v\big(\frac{k}{q_1^i}, k_4q_4^{i-1}\sigma^{i-1}, k_5q_5^{i-1}\sigma^{i-1}\big), \tag{3.44}$$

(d). $v(k, k_4, k_5) = \frac{1}{\gamma^m} v\big(kq_1^m, k_4q_4^m\sigma^m, k_5q_5^m\sigma^m\big) - \sum\limits_{i=1}^{m} \frac{\beta}{\gamma^i} v\big(kq_1^i, k_4q_4^{i-1}\sigma^i, k_5q_5^{i-1}\sigma^i\big)$

$$- \sum_{i=1}^{m} \tfrac{1}{\gamma^{i-1}}\Big[v\big(kq_1^{i+1}, k_4q_4^{i-1}\sigma^{i-1}, k_5q_5^{i-1}\sigma^{i-1}\big) + v\big(kq_2^{\pm}q_1^i, k_4q_4^{i-1}\sigma^{i-1}, k_5q_5^{i-1}\sigma^{i-1}\big)\Big]$$

$$- \sum_{i=1}^{m} \frac{6\alpha}{\gamma^{i-1}} v\big(kq_1^i, k_4q_4^{i-1}\sigma^{i-1}, k_5q_5^{i-1}\sigma^{i-1}\big). \tag{3.45}$$

Proof. The proof and verification are as similar as Theorem 3.5.2. □

Since heat flow is not instantaneous, the time needed to pass the heat energy from one point to another point is taken into account. Hence, we have introduced the delay factor in the heat equation model. Here, we used q-heat equation since logarithmic functions can be considered as solutions of q-heat equations. This is one of the significances of our book.

Chapter 4

Discrete Partial q-Heat Equation Models

4.1 Introduction

This chapter focuses on the formulation and corresponding solutions of the discrete partial q-heat equation as well as the discrete partial q-heat equation models with several variables. Being an application of difference operator, relevant formulae for finite and infinite series on polynomial and rational functions in number theory have been derived. This complex terrain of study finds its application in heat propagation within the given medium based on the Fourier law of conduction. It enables the optimal choice of material and gives us the knowledge about the nature of propagation of heat. The results are verified by MATLAB to validate the findings.

4.2 Preliminaries

Consider n-variable $k = (k_1, ..., k_n)$, shift value $\ell = (\ell_1, ..., \ell_n)$ and real valued function $u(k)$ defined on $\mathbb{R}^n$. The difference operator with shift value ℓ is defined as $\underset{\ell}{\Delta} u(k) = u(k+\ell) - u(k)$ and its inverse defined in (1.3) can be expressed as

$$v(k)|_{k-m\ell}^{k} = \overset{-1}{\underset{\ell}{\Delta}} u(k)|_{k-m\ell}^{k} = v(k) - v(k-m\ell). \tag{4.1}$$

Example 4.2.1. *If $u(k) = k_1k_2$, $\ell = (\ell_1, \ell_2)$, then $\underset{\ell}{\Delta} u(k) = (\ell_2k_2 + \ell_1k_2 + \ell_2\ell_2)$ gives $\overset{-1}{\underset{\ell}{\Delta}}(\ell_2k_2 + \ell_1k_2 + \ell_1\ell_2) = k_1k_2$. If $\ell_2 = 0$, then $\overset{-1}{\underset{\ell}{\Delta}}(\ell_1k_2) = k_1k_2$.*

Lemma 4.2.2. *If $E_i^{\ell_i}u(k) = u(k_1, k_2, ..., k_i + \ell_i, ..., k_n)$ for $i = 1, 2, ..., n$, then $1 + E^{\ell} = \underset{(\ell)}{\Delta}$, where $E^{\ell} = \prod_{i=1}^{n} E_i^{\ell_i}$.*

Proof. $\underset{(\ell)}{\Delta} u(k) = u(k_1+\ell_1, k_2+\ell_2, ..., k_n+\ell_n) - u(k) = E^{\ell}u(k) - u(k)$,

which gives $1 + \underset{(\ell)}{\Delta} = E^{\ell}$. $\square$

Corollary 4.2.3. *If $E^{r\ell} = E_1^{r\ell_1}E_2^{r\ell_2}...E_n^{r\ell_n}$, then $\overset{n}{\underset{\ell}{\Delta}} = (E^{\ell} - 1)^n = \sum_{r=0}^{n}(-1)^r nC_r E^{r\ell}$.*

Example 4.2.4. *From example 4.2.1, the relation (1.4) directly yields the relation $k_1k_2 - (k_1m\ell_1)(k_2 - m\ell_2) = \sum_{r=1}^{m}\{\ell_2(k_1 - r\ell_1) + \ell_1(k_2 - r\ell_2)\} + m\ell_1\ell_2$.*

Theorem 4.2.5. [31]*(Product Formula) If $u(k)$ and $v(k)$ are two real valued functions of n variables $k = (k_1, k_2, ..., k_n)$ and $\ell = (\ell_1, \ell_2, .., \ell_n)$ is n-shift value, then* $\overset{-1}{\underset{\ell}{\Delta}}\Big[u(k)v(k)\Big] = u(k)\overset{-1}{\underset{\ell}{\Delta}} v(k) - \overset{-1}{\underset{\ell}{\Delta}}\Big\{\overset{-1}{\underset{\ell}{\Delta}} v(k+\ell)\underset{\ell}{\Delta} u(k)\Big\}.$

Example 4.2.6. *Taking $u(k) = k_1k_2$ and $v(k) = a^{s(k_1+k_2)}$ in Theorem 4.2.5 gives*

$$\overset{-1}{\underset{\ell}{\Delta}}\left[(k_1k_2)a^{s(k_1+k_2)}\right] = (k_1k_2)\overset{-1}{\underset{\ell}{\Delta}} a^{s(k_1+k_2)} - \overset{-1}{\underset{\ell}{\Delta}}\left\{\overset{-1}{\underset{\ell}{\Delta}}\left[a^{s(k_1+k_2+\ell_1+\ell_2)}\underset{\ell}{\Delta} k_1k_2\right]\right\}.$$

4.3 Finite series formula on rational functions

In this section, we obtain finite series formula by equating closed and summation form solutions of equation (1.3) for a given rational function $u(k)$. Here, we use the polynomial factorial $k_\ell^{(n)} = k(k-\ell)(k-2\ell)...(k-(n-1)\ell)$, n is a positive integer. Recall the difference operator defined by $\underset{\ell}{\Delta} u(k) = u(k_1+\ell_1, k_2+\ell_2, \cdots, k_n+\ell_n) - u(k_1, k_2, \cdots, k_n)$. This is called partial difference operator if $n \geq 2$ and atleast one ℓ_i is zero but not all ℓ_i.

Lemma 4.3.1. *If $\ell = (\ell_1, \ell_2) \neq 0$ and $k_1, k_2 \neq 0$ are variables then we have*

$$\frac{1}{k_1k_2} = \overset{-1}{\underset{\ell}{\Delta}}\left\{\frac{\ell_1k_2 + \ell_2k_1 + \ell_1\ell_2}{(k_1+\ell_1)^{(2)}_{\ell_1}(k_2+\ell_2)^{(2)}_{\ell_2}}\right\}.$$

Proof. The proof follows from $\underset{\ell}{\Delta}\frac{1}{k_1k_2} = \frac{\ell_1k_2 + \ell_2k_1 + \ell_1\ell_2}{(k_1+\ell_1)^{(2)}_{\ell_1}(k_2+\ell_2)^{(2)}_{\ell_2}}$ and (1.4). □

Example 4.3.2. *If $\ell = (0, \ell_2, \ell_3) \neq 0$ and $\frac{1}{k_1k_2k_3} \neq 0$, (1.3) gives*

$$\underset{\ell}{\Delta}\frac{1}{k_1k_2k_3} = \frac{1}{k_1(k_2+\ell_2)(k_3+\ell_3)} - \frac{1}{k_1k_2k_3} = \frac{-[k_2\ell_3 + k_3\ell_2 + \ell_2\ell_3]}{k_1(k_2+\ell_2)^{(2)}_{\ell_2}(k_3+\ell_3)^{(2)}_{\ell_3}} \text{ and}$$

$$\overset{-1}{\underset{\ell}{\Delta}}\left[\frac{k_2\ell_3+k_3\ell_2+\ell_2\ell_3}{k_1(k_2+\ell_2)^{(2)}_{\ell_2}(k_3+\ell_3)^{(2)}_{\ell_3}}\right]=\frac{-1}{k_1k_2k_3}.$$

Theorem 4.3.3. *If k_1 and k_2 are not integer multiple of ℓ_1 and ℓ_2 respectively, then*

$$\sum_{r=1}^{m}\left\{\frac{\ell_1(k_2-r\ell_2)+\ell_2(k_1-r\ell_1)+\ell_1\ell_2}{(k_1+\ell_1-r\ell_1)^{(2)}_{\ell_1}(k_2+\ell_2-r\ell_2)^{(2)}_{\ell_2}}\right\}=\frac{1}{(k_1-m\ell_1)(k_2-m\ell_2)}-\frac{1}{k_1k_2}.$$

Proof. The proof follows from Lemma 4.3.1 and (1.4). □

Remark 4.3.4. *As $m\to\infty$, in Theorem 4.3.3, we arrive as the formula of backward infinite series* $\sum_{r=1}^{\infty}\left\{\frac{\ell_1(k_2-r\ell_2)+\ell_2(k_1-r\ell_1)+\ell_2\ell_1}{(k_1+\ell_1-r\ell_1)^{(2)}_{\ell_1}(k_2+\ell_2-r\ell_2)^{(2)}_{\ell_2}}\right\}=\frac{-1}{k_1k_2}.$

Lemma 4.3.5. *Let $(k_1)^{(2)}_{\ell_1}(k_2)^{(2)}_{\ell_2}\neq 0$ and $\ell=(\ell_1,\ell_2)$. The inverse difference of rational functions of polynomial factorial of two variables is given by*

$$\frac{-1}{2(k_1)^{(2)}_{\ell_1}(k_2)^{(2)}_{\ell_2}}=\overset{-1}{\underset{\ell}{\Delta}}\left\{\frac{\ell_1(k_2)+\ell_2(k_1)}{(k_1+\ell_1)^{(3)}_{\ell_1}(k_2+\ell_2)^{(3)}_{\ell_2}}\right\}.$$

Proof. By (1.3) and applying $\underset{\ell}{\Delta}$ on $\dfrac{-1}{2(k_1)^{(2)}_{\ell_1}(k_2)^{2}_{\ell_2}}$ completes the proof. □

Theorem 4.3.6. *If the denominators are non zero, and $m>0$ is integer, then*

$$\sum_{r=1}^{m}\left\{\frac{\ell_1(k_2-r\ell_2)+\ell_2(k_1-r\ell_1)}{(k_1+\ell_1-r\ell_1)^{(3)}_{\ell_1}(k_2+\ell_2-r\ell_2)^{(3)}_{\ell_2}}\right\}=\frac{1}{2(k_1-m_{\ell_1})^{(2)}_{\ell_1}(k_2-m_{\ell_2})^{(2)}_{\ell_2}}-\frac{1}{2(k_1)^{(2)}_{\ell_1}(k_2)^{(2)}_{\ell_2}}.$$

Proof. The proof follows by (1.4) and Lemma 4.3.5. □

Remark 4.3.7. *As $m\to\infty$, infinite series backward formula in polynomial factorial is* $\sum_{r=1}^{m}\left\{\frac{\ell_1(k_2-r\ell_2)+\ell_2(k_1-r\ell_1)}{(k_1+\ell_1-r\ell_1)^{(3)}_{\ell_1}(k_2+\ell_2-r\ell_2)^{(3)}_{\ell_2}}\right\}=\frac{-1}{2(k_1)^{(2)}_{\ell_1}(k_2)^{(2)}_{\ell_2}}.$

Lemma 4.3.8. *If $\ell=(0,\ell_2,\ell_3)$, $k_1\neq 0$, then the inverse difference of rational functions of three variables is given by*

$$-\frac{1}{k_1k_2k_3}=\overset{-1}{\underset{\ell}{\Delta}}\left\{\frac{\ell_3k_2+\ell_2k_3+\ell_2\ell_3}{k_1(k_2+\ell_2)^{(2)}_{\ell_2}(k_3+\ell_3)^{(2)}_{\ell_3}}\right\}.$$

Proof. The proof follows by taking $\overset{-1}{\underset{\ell}{\Delta}}$ on both sides. $\square$

Theorem 4.3.9. *If k_2, k_3 are not integer multiple of ℓ_2, ℓ_3, k_3 respectively, then*
$$\sum_{r=1}^{m}\left\{\frac{\ell_3(k_2-r\ell_2)+\ell_2(k_3-r\ell_3)+\ell_2\ell_3}{k_1(k_2+\ell_2-r\ell_2)^{(2)}_{\ell_2}(k_3+\ell_3-r\ell_3)^{(2)}_{\ell_3}}\right\}=\frac{1}{k_1(k_2-m\ell_2)(k_3-m\ell_3)}-\frac{1}{k_1k_2k_3}.$$

Proof. Lemma 4.3.8 and (1.4) yield the finite summation formula. $\square$

4.4 Infinite series formula on rational functions

In this section, we derive the forward infinite series formula for certain class of rational functions. Also we use infinite series solution in heat equation model.

Lemma 4.4.1. *The infinite inverse principle of $u(k)$ with respect to $\underset{\ell}{\Delta}$ is given by* $\sum_{r=0}^{\infty}u(k+r\ell)=-\overset{-1}{\underset{\ell}{\Delta}}u(k)\Big|_{k}^{\infty}$ *if the series is convergent i.e.,* $\lim_{m\to\infty}\overset{-1}{\underset{\ell}{\Delta}}u(k+m\ell)=0.$

Proof. The proof follows from $\lim_{m\to\infty}\overset{-1}{\underset{\ell}{\Delta}}u(k+m\ell)=0$, (1.3) and
$\underset{\ell}{\Delta}\left\{\sum_{r=0}^{\infty}u(k+r\ell)\right\}=u(k).$ $\square$

Theorem 4.4.2. *Assume that $\ell_1,\ell_2>0$ and k_1,k_2 are not an integer multiple of ℓ_1 and ℓ_2 respectively. Then an infinite series and closed form solution of the equation*
$$\sum_{r=0}^{\infty}\left\{\frac{\ell_1(k_2+r\ell_2)+\ell_2(k_1+r\ell_1)+\ell_1\ell_2}{(k_1+\ell_1+r\ell_1)^{(2)}_{\ell_1}(k_2+\ell_2+r\ell_2)^{(2)}_{\ell_2}}\right\}=\frac{1}{k_1k_2}.$$

Proof. Lemma 4.4.1 and 4.3.1 complete the proof by taking $u(k)=\dfrac{1}{k_1k_2}$. $\square$

The following theorem is the forward infinite series formula of reciprocal of product of polynomial factorials.

Theorem 4.4.3. *If k_1 is not a multiple of ℓ_1, k_2 is not a multiple of ℓ_2 then* $\sum_{r=0}^{\infty}\left\{\dfrac{\ell_1(k_2+r\ell_2)+\ell_2(k_1+r\ell_1)}{(k_1+\ell_1+r\ell_1)^{(3)}_{\ell_1}(k_2+\ell_2+r\ell_2)^{(3)}_{\ell_2}}\right\}=\dfrac{1}{2(k_1)^{(2)}_{\ell_1}(k_2)^{(2)}_{\ell_2}}.$

Proof. Lemmas 4.4.1 and 4.3.5 generate the forward infinite series formula. □

Theorems 4.4.2 and 4.4.3 are the infinite series formula for certain class of rational functions.

4.5 Formation of partial q-heat equation

For $q=(q_1,\ q_2,\ldots,\ q_n)$, the generalized q-difference operator is defined as

$$\underset{q}{\Delta} v(k)=v(k_1q_1,\ k_2q_2,\ldots,\ k_nq_n)-v(k_1,\ k_2,\ldots,\ k_n), q_i>0, \tag{4.2}$$

where $k=(k_1,\ k_2,\ldots,\ k_n)\in R^n$, and $v(k):R^n\to R$ is a real valued function. The operator $\underset{q}{\Delta}$ becomes partial q-difference operator if some $q_i=1$.

Consider the notations in section 2.5. By Newton cooling law, the discrete q-heat equation with rate constants α and β is

$$\underset{(1,q_2)}{\Delta} v(k_1,k_2)=\alpha \underset{(q_1^{-1},1)}{\Delta} v(k_1,k_2)+\beta \underset{(q_1,1)}{\Delta} v(k_1,k_2), \tag{4.3}$$

where the operators $\underset{(1,q_2)}{\Delta}$, $\underset{(q_1^{-1},1)}{\Delta}$ and $\underset{(q_1,1)}{\Delta}$ are as given in (4.2), having $n=2$. Here, we extend the theory developed in section 2.5 with two rate constants.

4.5.1 Solution of q-heat equation of long rod

In this section, we derive inverse principle for q-partial difference operator and provide several kinds of solution of partial q-heat equation (4.3). Here, we assume that $q_i \neq 0$, $k_1, k_2 \in (-\infty, \infty)$ and m is a positive integer.

Theorem 4.5.1. *If* $\underset{(q_1^{-1},1)}{\Delta} v(k_1,k_2) = \underset{q_1^{-1}}{u}(k_1,k_2)$ *and* $\underset{(q_1,1)}{\Delta} v(k_1,k_2) = \underset{q_1}{u}(k_1,k_2)$ *are known functions, then the q-heat equation* (4.3) *has a numerical solution of the form*

$$v(k_1,k_2) - v(k_1,k_2q_2^{-m}) = \sum_{r=1}^{m} \left\{ \alpha \underset{q_1^{-1}}{u}(k_1,k_2q_2^{-r}) + \beta \underset{q_1^{-1}}{u}(k_1,k_2q_2^{-r}) \right\}. \tag{4.4}$$

Proof. The proof follows from (4.3) and applying the inverse principle (1.4). □

Theorem 4.5.2. *If* $\alpha = \beta$, *then* $v(k_1,k_2)$ *of* (4.3) *satisfies the following relations:*

(i) $v(k_1,k_2) = \dfrac{1}{(1-2\alpha)^m} v(k_1,k_2q_2^m)$

$$-\sum_{r=1}^{m} \frac{\alpha}{(1-2\alpha)^r} \left(v(k_1q_1^{-1},k_2q_2^{r-1}) + v(k_1q_1,k_2q_2^{r-1}) \right), \tag{4.5}$$

(ii) $v(k_1,k_2) = \dfrac{(1-2\alpha)^2}{(1-\alpha)(1-3\alpha)} \left\{ \sum\limits_{r=1}^{m} \dfrac{(-1)^{r-1}\alpha^{r-1}}{(1-2\alpha)^r} v(k_1q_1^{1-r},k_2q_2) \right.$

$$\left. + \sum_{r=1 r\neq 2}^{m} \frac{(-1)^r\alpha^r}{(1-2\alpha)^r} v(k_1q_1^{2-r},k_2) + \frac{(-1)^m\alpha^m}{(1-2\alpha)^m} v(k_1q_1^{-m},k_2) \right\}, \tag{4.6}$$

and (iii) $v(k_1,k_2) = \dfrac{(1-2\alpha)^2}{(1-\alpha)(1-3\alpha)} \left\{ \sum\limits_{r=1}^{m} \dfrac{(-1)^{r-1}\alpha^{r-1}}{(1-2\alpha)^r} v\Big(k_1q_1^{r-1},k_2q_2\Big) \right.$

$$\left. + \sum_{r=1 r\neq 2}^{m} \frac{(-1)^r\alpha^r}{(1-2\alpha)^r} v\Big(k_1q_1^{r-2},k_2\Big) + \frac{(-1)^m\alpha^m}{(1-2\alpha)^m} v\Big(k_1q_1^{m},k_2\Big) \right\}. \tag{4.7}$$

Proof. Since $\alpha = \beta$, from (4.3), we obtain

$$v(k_1, k_2) = \frac{1}{1-2\alpha} v(k_1, k_2 q_2) - \frac{\alpha}{1-2\alpha} v(k_1 q_1^{-1}, k_2) - \frac{\alpha}{1-2\alpha} v(k_1 q_1, k_2). \tag{4.8}$$

(i) Replacing k_2 by $k_2 q_2$ in (4.8) and continuing the same process, we get (4.5),

(ii) replacing k_1 by $k_1 q_1^{-1}$ in (4.8), yields (4.6) and

(iii) replacing k_1 by $k_1 q_1$ in (4.8), gives (4.7). □

Remark 4.5.3. *If* $\alpha = \dfrac{q_1(q_2-1)}{(1-q_1)^2}$, $q_i \neq 1$, *then* $v(k_1, k_2) = k_1 k_2$ *is a closed form solution of the* q*-heat equation* (4.3)*. Also,* $v(k_1, k_2) = k_1 k_2$ *satisfies equations* (4.5) *to* (4.8)*.*

Example 4.5.4. *For* $(k_1,\ k_2) \in R^2$, *if* $1 \neq 2\alpha$, *then* $v(k_1, k_2) = k_1 k_2$ *in* (4.5) *gives*

$$k_1 k_2 = \frac{k_1 k_2 q_2^m}{(1-2\alpha)^m} - \sum_{r=1}^{m} \frac{\alpha}{(1-2\alpha)^r} \left(k_1 k_2 q_2^{r-1} q_1^{-1} + k_1 q_1 k_2 q_2^{r-1} \right). \tag{4.9}$$

For numerical verification of (4.9)*, when* $k_1 = 2$, $k_2 = 3$, $q_1 = 2$ $q_2 = 4$, $\alpha = 6$ *and* $m = 50$ *the MATLAB coding is given below:*

$6 = (6. * (4. \wedge 50))./((1-2. * (6)). \wedge (50)) - symsum(((6)./(1-2. * (6)). \wedge r). * (((6. * (4. \wedge (r-1)))./2) + (12. * (4. \wedge (r-1)))), r, 1, 50).$

Theorem 4.5.5. *If* $\beta = -\alpha$, *then* $v(k_1, k_2)$ *of* (4.3) *satisfies*

$$v(k_1, k_2) = v(k_1, k_2 q_2^m) + \alpha \sum_{r=0}^{m} \left(v(k_1 q_1, k_2 q_2^r) - v(k_1 q_1^{-1}, k_2 q_2^r) \right). \tag{4.10}$$

Proof. The proof follows from (4.3), $\beta = -\alpha$ and replacing k_2 by $k_2 q_2$ repeatedly in

$$v(k_1, k_2) = v(k_1, k_2 q_2) - \alpha \left\{ v(\frac{k_1}{q_1}, k_2) - v(k_1 q_1, k_2) \right\}. \tag{4.11}$$

$\square$

Example 4.5.6. *The function* $v(k_1,k_2) = \log(k_1k_2)$ *is a solution of* (4.3) *when* $\alpha = -\dfrac{\log q_2}{2\log q_1}, q_1 \neq 1, q_2 \neq 0$ *and* $\beta = -\alpha$. *Taking* $v(k_1,k_2) = \log(k_1k_2)$ *in* (4.10),

$$\log(k_1k_2) = \log(k_1k_2q_2^m) + \alpha\sum_{r=0}^{m}\left(\log(k_1q_1k_2q_2^r) - \log(k_1k_2q_2^rq_1^{-1})\right). \tag{4.12}$$

The formula (4.12) *is verified by taking* $k_1 = 2,\ k_2 = 3,\ q_1 = 2,\ q_2 = 4,\ \alpha = -1,$ $m = 150$ *and the MATLAB coding is given by*

$\log(6) = \log(6.*(4.\wedge(150))) - symsum(\log(12.*(4.\wedge(r-1))) - \log(3.*(4.\wedge(r-1))), r, 1, 150)$.

4.5.2 Solution of q-heat equation of thin plate

As in the case of rod, if $v(k_1,k_2,k_3)$ is the temperature at the position (k_1,k_2) at time k_3, then the q-heat equation of thin plate takes the form

$$\underset{(1,1,q_3)}{\Delta} v(k_1,k_2,k_3) = \alpha \underset{(q_1^{-1},q_2^{-1},1)}{\Delta} v(k_1,k_2,k_3) + \beta \underset{(q_1,q_2,1)}{\Delta} v(k_1,k_2,k_3). \tag{4.13}$$

Theorem 4.5.7. *When* $\alpha = \beta$, $v(k_1,k_2,k_3)$ *of* (4.13) *satisfies*

(i). $v(k_1,k_2,k_3) = \dfrac{1}{(1-2\alpha)^m}v(k_1,k_2,k_3q_3^m)$

$$-\sum_{r=1}^{m}\frac{\alpha}{(1-2\alpha)^r}\left(v(k_1q_1^{-1},k_2q_2^{-1},k_3q_3^{r-1}) + v(k_1q_1,k_2q_2,k_3q_3^{r-1})\right). \tag{4.14}$$

(ii). $v(k_1,k_2,k_3) = \dfrac{(1-2\alpha)^2}{(1-\alpha)(1-3\alpha)}\left\{\sum\limits_{r=1}^{m}\dfrac{(-1)^{r-1}\alpha^{r-1}}{(1-2\alpha)^r}v(k_1q_1^{1-r},k_2q_2^{1-r},k_3q_3)\right.$

$$\left.+\sum_{r=1 r\neq 2}^{m}\frac{(-1)^r\alpha^r}{(1-2\alpha)^r}v(k_1q_1^{2-r},k_2q_2^{2-r},k_3) + \frac{(-1)^m\alpha^m}{(1-2\alpha)^m}v(k_1q_1^{-m},k_2q_2^{-m},k_3)\right\}. \tag{4.15}$$

(iii). $v(k_1,k_2,k_3) = \dfrac{(1-2\alpha)^2}{(1-\alpha)(1-3\alpha)} \left\{ \sum_{r=1}^{m} \dfrac{(-1)^{r-1}\alpha^{r-1}}{(1-2\alpha)^r} v(k_1 q_1^{r-1}, k_2 q_2^{r-1}, k_3 q_3) \right.$

$$\left. + \sum_{r=1 r\neq 2}^{m} \frac{(-1)^r \alpha^r}{(1-2\alpha)^r} v(k_1 q_1^{r-2}, k_2 q_2^{r-2}, k_3) + \frac{(-1)^m \alpha^m}{(1-2\alpha)^m} v(k_1 q_1^m, k_2 q_2^m, k_3) \right\}. \quad (4.16)$$

Proof. From (4.13) and rearranging the terms, we obtain

$$v(k_1,k_2,k_3) = \frac{1}{1-2\alpha} v(k_1,k_2,k_3q_3) - \frac{\alpha}{1-2\alpha}[v(k_1q_1^{-1}, k_2q_2^{-1}, k_3) + v(k_1q_1, k_2q_2, k_3)]. \quad (4.17)$$

Replacing k_3 by k_3q_3 repeatedly, k_1 by $k_1q_1^{-1}$ and k_2 by $k_2q_2^{-1}$ and k_1 by k_1q_1 and k_2 by k_2q_2 in (4.17) yields (i),(ii) and (iii). □

Remark 4.5.8. *Taking $\beta = -\alpha$ in (4.13) yields*

$$\underset{(1,1,q_3)}{\Delta} v(k_1,k_2,k_3) = \alpha \left\{ v(\frac{k_1}{q_1}, \frac{k_2}{q_2}, k_3) - v(k_1q_1, k_2q_2, k_3) \right\}, \quad (4.18)$$

and hence we arrive

$$v(k_1,k_2,k_3) = v(k_1,k_2,k_3q_3^m) + \alpha \sum_{r=1}^{m} \left(v(k_1q_1, k_2q_2, k_3q_3^{r-1}) - v(\frac{k_1}{q_1}, \frac{k_2}{q_2}, k_3q_3^{r-1}) \right). \quad (4.19)$$

Example 4.5.9. *If $k_1 \neq 0$, $q_i \neq 0, 1$, $\alpha = -\dfrac{\log q_3}{2\log q_1q_2}$ and $\beta = -\alpha$, from (4.19),*

$$\log k_1k_2k_3 = \log k_1k_2k_3q_3^m + \alpha \sum_{r=1}^{m} \left(\log k_1q_1k_2q_2k_3q_3^{r-1} - \log \frac{k_1k_2k_3q_3^{r-1}}{q_1q_2} \right),$$

is a solution of discrete q-heat equation (4.13).

4.5.3 Solution of q-heat equation for medium

A linear generalized partial difference equation is of the form

$$\underset{(q)}{\Delta} v(k) = u(k), \quad (4.20)$$

where $\underset{(q)}{\Delta}$ is as given in (1.3), $q_i = 1$ for some i and $u(k) : \mathbb{R}^n \to \mathbb{R}$ is a given function. A function $v(k) : \mathbb{R}^n \to \mathbb{R}$ satisfying (4.20) is called a solution of the equation (4.20). The equation (4.20) has a numerical solution of the form,

$$v(k) - v(\frac{k}{q^m}) = \sum_{r=1}^{m} u(\frac{k}{q^r}), \tag{4.21}$$

where $\frac{k}{q^r} = (\frac{k_1}{{q_1}^r}, \frac{k_2}{{q_2}^r}, ..., \frac{k_m}{{q_m}^r})$, m is any positive integer. Relation (4.21) is the basic inverse principle with respect to $\underset{(q)}{\Delta}$ ([8, 27, 31]).

For example, the basic inverse principle with respect to $\underset{(1,q_2)}{\Delta}$ is given by

$$v(k_1, k_2) - v(k_1, \frac{k_2}{{q_2}^m}) = \sum_{r=1}^{m} u\left(k_1, \frac{k_2}{{q_2}^r}\right), \tag{4.22}$$

where $v(k_1, k_2) = \underset{(1,q_2)}{\Delta^{-1}} u(k_1, k_2)$. From the theory of generalized difference equation, we have two types of solutions to (4.20), namely closed form and summation form solutions [8, 27, 31]. Similarly, the partial difference equation (4.20) has two types of solutions.

Consider the notations in heat flows. The proportional amount of heat flows from left to right at $(k_1, k_2, k_3, k_4, k_5)$ is $\underset{(\frac{1}{q_1},1,1)}{\Delta} v(k)$, right to left is $\underset{(q_1,1,1)}{\Delta} v(k)$, top to bottom is $\underset{(1,q_2,1)}{\Delta} v(k)$, bottom to top is $\underset{(1,\frac{1}{q_2},1)}{\Delta} v(k)$, rear to back is $\underset{(1,1,q_3)}{\Delta} v(k)$, back to rear is $\underset{(1,1,\frac{1}{q_3})}{\Delta} v(k)$. By Fourier law of cooling, the heat equation for medium in $\Re^3$ is

$$\underset{(q_4,q_5)}{\Delta} v(k) = \gamma \underset{(q^{\pm}_{(1,2,3)})}{\Delta} v(k), \tag{4.23}$$

where $\underset{(q^{\pm}(1,2,3))}{\Delta} = \underset{(q_1)}{\Delta} + \underset{(\frac{1}{q_1})}{\Delta} + \underset{(q_2)}{\Delta} + \underset{(\frac{1}{q_2})}{\Delta} + \underset{(q_3)}{\Delta} + \underset{(\frac{1}{q_3})}{\Delta}$ and $k = (k_1, k_2, k_3, k_4, k_5)$.

Theorem 4.5.10. *Assume that* $v(k_1, k_2, k_3, \frac{k_4}{{q_4}^m}, \frac{k_5}{{q_5}^m})$ *and the partial differences* $\underset{(q^{\pm}_{(1,2,3)})}{\Delta} v(k) = \underset{(q^{\pm}_{(1,2,3)})}{u}(k)$ *are known functions. Then, the partial q-heat equation* (4.23) *has a solution of the form*

$$v(k) = v(k_1, k_2, k_3, \frac{k_4}{{q_4}^m}, \frac{k_5}{{q_5}^m}) + \gamma \sum_{r=1}^{m} \underset{(q^{\pm}_{(1,2,3)})}{u}(k_1, k_2, k_3, \frac{k_4}{{q_4}^r}, \frac{k_5}{{q_5}^r}). \tag{4.24}$$

Proof. Taking $\underset{(q^{\pm}_{(1,2,3)})}{\Delta} v(k) = \underset{(q^{\pm}_{(1,2,3)})}{u}(k)$ in (4.23), we get

$$v(k) = \gamma \underset{(q_4,q_5)}{\Delta^{-1}} \underset{(q^{\pm}_{(1,2,3)})}{u}(k). \tag{4.25}$$

The proof follows by applying inverse principle (4.22) in (4.25). □

In the Theorem 4.5.11, we use the following notations:

$$\begin{aligned} v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), *, *) = v(k_1q_1, k_2, k_3, *, *) + v(\frac{k_1}{q_1}, k_2, k_3, *, *) \\ + v(k_1, k_2q_2, k_3, *, *) + v(k_1, \frac{k_2}{q_2}, k_3, *, *) \\ + v(k_1, k_2, k_3q_3, *, *) + v(k_1, k_2, \frac{k_3}{q_3}, *, *). \end{aligned}$$

$$\begin{aligned} v(*, k_{(2,3),*,*}(*, q^{\pm}_{(2,3)}), *, *) = v(*, k_2q_2, k_3, *, *) + v(*, \frac{k_2}{q_2}, k_3, *, *) \\ + v(*, k_2, k_3q_3, *, *) + v(*, k_2, \frac{k_3}{q_3}, *, *). \end{aligned}$$

Theorem 4.5.11. *If* $v(k)$ *is a solution of the equation* (4.23) *and* m *is a positive integer, then the following relations are equivalent:*

(a). $v(k) = (1-6\gamma)^m v(k_1, k_2, k_3, \frac{k_4}{{q_4}^m}, \frac{k_5}{{q_5}^m})$

$$+ \sum_{r=0}^{m-1} \gamma(1-6\gamma)^r \left[v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), \frac{k_4}{q_4^{(r+1)}}, \frac{k_5}{q_5^{(r+1)}}) \right], \tag{4.26}$$

$$\text{(b). } v(k) = \frac{1}{(1-6\gamma)^m} v(k_1, k_2, k_3, k_4 {q_4}^m, k_5 {q_5}^m) - \sum_{r=1}^{m} \frac{\gamma}{(1-6\gamma)^r} \Big[v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), k_4 {q_4}^{(r-1)}, k_5 {q_5}^{(r-1)}) \Big], \tag{4.27}$$

$$\text{(c). } v(k) = \frac{1}{\gamma^m} v(\frac{k_1}{{q_1}^m}, k_2, k_3, k_4 {q_4}^m, k_5 {q_5}^m) - \sum_{r=1}^{m} \frac{1-6\gamma}{\gamma^r} v(\frac{k_1}{{q_1}^r}, k_2, k_3, k_4 {q_4}^{(r-1)}, k_5 {q_5}^{(r-1)}) - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v\left(\frac{k_1}{{q_1}^{(r+1)}}, k_{(2,3)}(q^{\pm}_{(2,3)}), k_4 {q_4}^r, k_5 {q_5}^r \right), \tag{4.28}$$

$$\text{(d). } v(k) = \frac{1}{\gamma^m} v(k_1 {q_1}^m, k_2, k_3, k_4 {q_4}^m, k_5 {q_5}^m) - \sum_{r=1}^{m} \frac{1-6\gamma}{\gamma^r} v(k_1 {q_1}^r, k_2, k_3, k_4 {q_4}^{(r-1)}, k_5 {q_5}^{(r-1)}) - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 {q_1}^{(r+1)}, k_{(2,3)}(q^{\pm}_{(2,3)}), k_4 {q_4}^r, k_5 {q_5}^r). \tag{4.29}$$

Proof. From (4.23) and (1.3), we arrive

(i). $v(k) = (1-6\gamma) v\left(k_1, k_2, k_3, \frac{k_4}{q_4}, \frac{k_5}{q_5} \right) + \gamma [v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), \frac{k_4}{q_4}, \frac{k_5}{q_5})]$,

(ii). $v(k) = \frac{1}{(1-6\gamma)} v(k_1, k_2, k_3, k_4 q_4, k_5 q_5) - \frac{\gamma}{(1-6\gamma)} [v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), k_4, k_5)]$,

(iii). $v(k) = \frac{1}{\gamma} v(\frac{k_1}{q_1}, k_2, k_3, k_4 q_4, k_5 q_5) - \frac{1-6\gamma}{\gamma} v(\frac{k_1}{q_1}, k_2, k_3, k_4, k_5) - v(\frac{k_1}{{q_1}^2}, k_2, k_3, k_4, k_5) - v(\frac{k_1}{q_1}, k_{(2,3)}(q^{\pm}_{(2,3)}), k_4, k_5)$ and

(iv). $v(k) = \frac{1}{\gamma} v(k_1 q_1, k_2, k_3, k_4 q_4, k_5 q_5) - \frac{1-6\gamma}{\gamma} v(k_1 q_1, k_2, k_3, k_4, k_5) - v(k_1 {q_1}^2, k_2, k_3, k_4, k_5) - v(k_1 q_1, k_{(2,3)}(q^{\pm}_{(2,3)}), k_4, k_5)$.

Now the proof of (a), (b), (c) and (d) follows by replacing

k_4 and k_5 by $\frac{k_4}{q_4}, \frac{k_4}{q_4^2}, ..., \frac{k_4}{q_4^m}$ and $\frac{k_5}{q_5}, \frac{k_5}{q_5^2}, ..., \frac{k_5}{q_5^m}$ in (i)

k_4 and k_5 by $k_4 q_4, k_4 q_4^2, ..., k_4 q_4^m$ and $k_5 q_5, k_5 q_5^2, ..., k_5 q_5^m$ in (ii)

k_1,k_4 and k_5 by $\frac{k_4}{q_4}, \frac{k_1}{q_1^2}, ..., \frac{k_1}{q_1^m}, k_4q_4, k_4q_4^2, ..., k_4q_4^m$ and $k_5q_5, k_5q_5^2, ..., k_5q_5^m$ in (iii) k_1, k_4 and k_5 by $k_1q_1, k_1q_1^2, ..., k_1q_1^m, k_4q_4, k_4q_4^2, ..., k_4q_4^m$ and $k_5q_5, k_5q_5^2, ..., k_5q_5^m$ in (iv) respectively. □

Example 4.5.12. *The following example shows that the diffusion of medium in three dimensional system can be identified if the solution $v(k_1, k_2, k_3, k_4, k_5)$ of (4.23) is known and vice versa. Suppose that $v(k_1, k_2, k_3, k_4, k_5) = k_1k_2k_3k_4k_5$ is a closed form solution of (4.23), then we have $\underset{(q_4,q_5)}{\Delta} k_1k_2k_3k_4k_5 = \gamma\left[\underset{(q^{\pm}_{(1,2,3)})}{\Delta} k_1k_2k_3k_4k_5\right]$, which yields $k_1k_2k_3k_4k_5(q_4q_5 - 1) = \gamma\big[k_1k_2k_3k_4k_5(q_1 + \frac{1}{q_1} + q_2 + \frac{1}{q_2} + q_3 + \frac{1}{q_3} - 6)\big]$. Cancelling $k_1k_2k_3k_4k_5$ on both sides gives*

$$\gamma = \frac{q_4q_5 - 1}{q_1 + \frac{1}{q_1} + q_2 + \frac{1}{q_2} + q_3 + \frac{1}{q_3} - 6}. \tag{4.30}$$

For numerical verification, if we assume that $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5$, $q_1 = 1, q_2 = 2, q_3 = 3, q_4 = 4, q_5 = 5, m = 2$ then $v(k_1, k_2, k_3, k_4, k_5) = 120$, $\gamma = \frac{4*5-1}{1+\frac{1}{1}+2+\frac{1}{2}+3+\frac{1}{3}-6}$, LHS and RHS of (a) of Theorem 4.5.11. is given below respectively.

$$120 = 1122.964462 + 487.0909103 - 1490.055371 \Rightarrow 120 = 120.$$

LHS and RHS of (b), (c), (d) of Theorem (4.5.11) are as similar as (a).

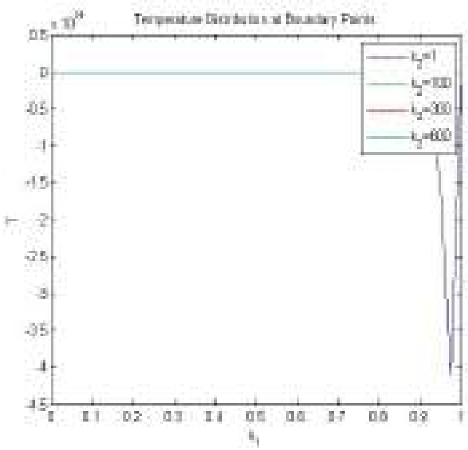
For matlab coding, if we assume that $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5, q_1 = 1$, $q_2 = 2, q_3 = 3, q_4 = 4, q_5 = 5, m = 5$ then we have

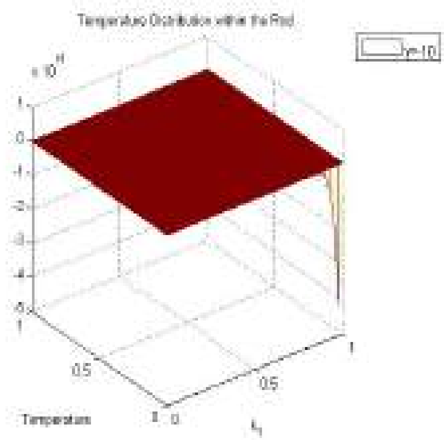
$1.*2.*3.*4.*5 = (1-6.*(10.36363636)).\wedge(5).*(6.*(4./((4).\wedge 5)).*(5./((5).\wedge 5)))+symsum((10.36363636).*(1-6.*(10.36363636)).\wedge r.*((6.*(4./(4.\wedge(r+1))).*$

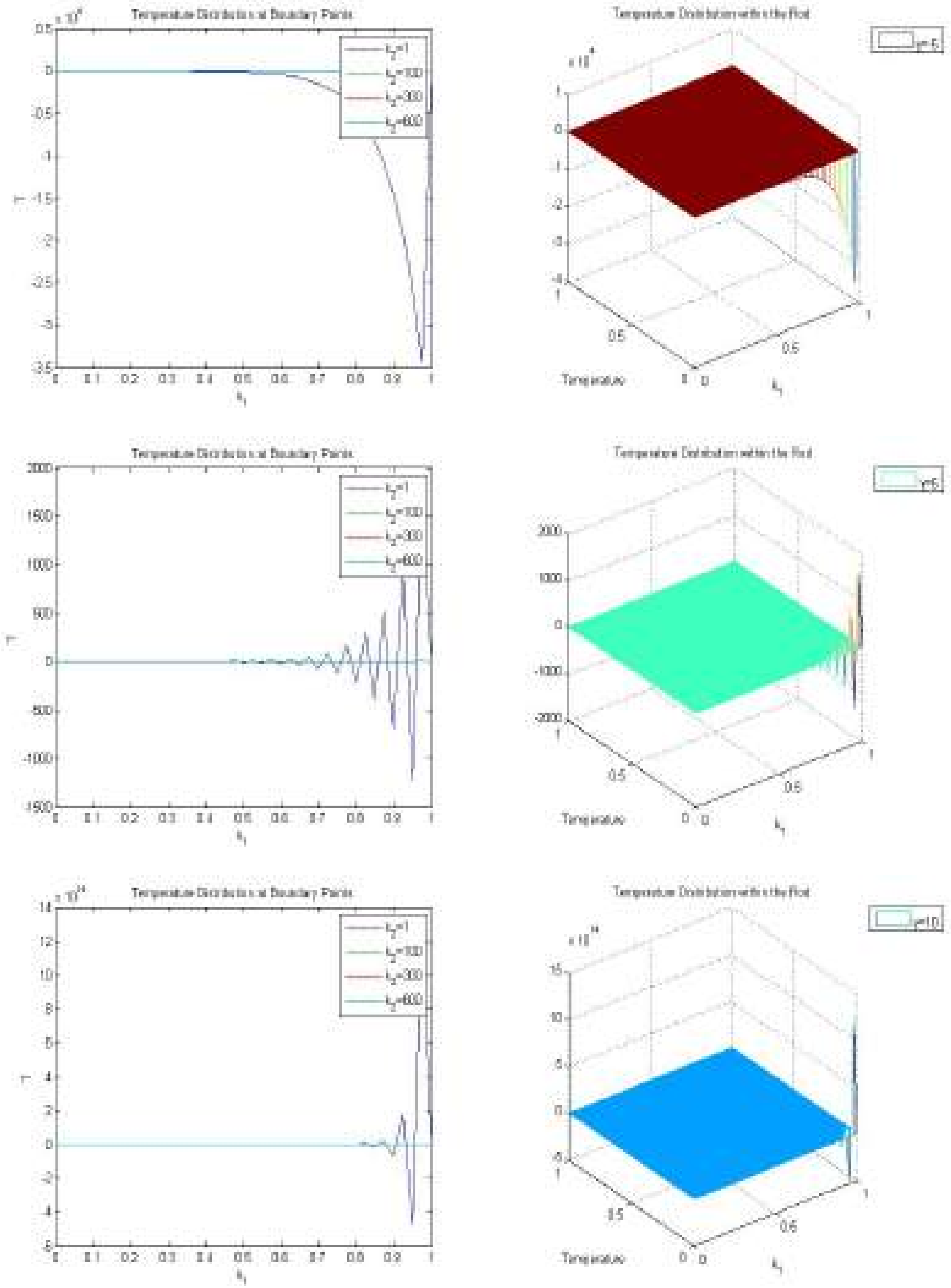
$(5./(5.\wedge(r+1))))+(6.*(4./(4.\wedge(r+1))).*(5./(5.\wedge(r+1))))+(12.*(4./(4.\wedge(r+1))).*(5./(5.\wedge(r+1))))+(3.*(4./(4.\wedge(r+1))).*(5./(5.\wedge(r+1))))+(18.*(4./(4.\wedge(r+1))).*(5./(5.\wedge(r+1))))+(2.*(4./(4.\wedge(r+1))).*(5./(5.\wedge(r+1))))),r,0,4)$

4.6 Logarithmic solutions in heat flows

To analyze the logarithmic solution of the q-heat equation (4.3), the following diagrams are obtained by taking $q_1 = \dfrac{1}{40}$, $q_2 = \dfrac{1}{1500}$, $\beta = -\alpha$ and $m = 1$ in (4.12) and boundary values $v(k_1,1)=\log(q_1+1)$, $v(1,k_2)=0, v(41,k_2) = 3.6889$.







From the diagram, we find that heat flows after certain stage is constant. Through our research, the saturated point can be identified easily using MATLAB.

4.7 Partial alpha-beta heat equation models

Let $\beta \neq 0$, $\ell = (\ell_1, \ell_2, \ell_3, ..., \ell_n) \neq 0$ and $v(k)$ be a real valued function of n variables $k = (k_1, k_2, k_3, ..., k_n)$. The β-difference operator on $v(k)$ is defined by

$$\underset{\beta(\ell)}{\Delta} v(k) = v(k_1 + \ell_1, k_2 + \ell_2, ..., k_n + \ell_n) - \beta v(k_1, k_2, ..., k_n). \tag{4.31}$$

This operator $\underset{\beta(\ell)}{\Delta}$ becomes partial β-difference operator if some $\ell_i = 0$. For a given function $u(k)$, a first order linear generalized partial β-difference equation is,

$$\underset{\beta(\ell)}{\Delta} v(k) = u(k), \tag{4.32}$$

has a numerical solution of the form

$$v(k) - \beta^m v(k - m\ell) = \sum_{r=1}^{m} \beta^{r-1} u(k - r\ell) = \underset{\beta(\ell)}{\overset{-1}{\Delta}} u(k)\Big|_{k-m\ell}^{k} \tag{4.33}$$

where $k - r\ell = (k_1 - r\ell_1, k_2 - r\ell_2, ..., k_m - r\ell_m)$, m is any positive integer. Relation (4.33) is the basic inverse principle with respect to $\underset{\beta(\ell)}{\Delta}$ [8]. Here, we apply the alpha-beta partial difference equation $\underset{\beta(\ell)}{\Delta} v(k) = \gamma \underset{\alpha(\pm\ell)}{\Delta} v(k)$ in heat flows.

4.7.1 Alpha-beta heat equation of long rod

With the heat equation parameters from (4.31) and Newton law of cooling, the discrete heat equation of rod is taken by

$$\underset{\beta(0,\ell_2)}{\Delta} v(k_1, k_2) = \gamma \underset{\alpha(\pm\ell_1,0)}{\Delta} v(k_1, k_2), \tag{4.34}$$

where $\underset{\alpha(\pm\ell_1,0)}{\Delta} = \underset{\alpha(\ell_1,0)}{\Delta} + \underset{\alpha(-\ell_1,0)}{\Delta}$.

Theorem 4.7.1. *Assume that there exists a positive integer m, and a real number $\ell_2 > 0$ such that $v(k_1, k_2 - m\ell_2)$ and $\underset{\alpha(\pm\ell_1)}{\Delta} v(k_1, k_2) = \underset{\alpha(\pm\ell_1)}{u}(k_1, k_2)$ are known. Then the heat equation (4.34) has a solution $v(k_1, k_2)$ of the form*

$$v(k_1, k_2) = \beta^m v(k_1, k_2 - m\ell_2) + \gamma \sum_{r=1}^{m} \beta^{r-1} \underset{\alpha(\pm\ell_1)}{u}(k_1, k_2 - r\ell_2). \tag{4.35}$$

Proof. By representing $\underset{\alpha(\pm\ell_1,0)}{\Delta} v(k_1, k_2) = \underset{\alpha(\pm\ell_1)}{u}(k_1, k_2)$, from (4.33) and (4.34),

$$v(k_1, k_2) - \beta^m v(k_1, k_2 - m\ell_2) = \gamma \underset{\beta(0,\ell_2)}{\overset{-1}{\Delta}} \underset{\alpha(\pm\ell_1)}{u}(k_1, k_2)\Big|_{k-m\ell}^{k} \tag{4.36}$$

which yields (4.35). $\square$

Example 4.7.2. *If $\gamma = \dfrac{e^{\ell_2} - \beta}{2(1-\alpha)}, \alpha \neq 1$, then $v(k_1, k_2) = k_1 e^{k_2}$ is a solution of (4.34). Hence $v(k_1, k_2) = k_1 e^{k_2}$ with $\underset{\alpha(\pm\ell_1,0)}{u}(k_1, k_2) = 2(1-\alpha)k_1 e^{k_2}$ satisfies (4.35). Similarly if, $\gamma = \dfrac{1+\beta}{2(1-\alpha)}, \alpha \neq 1$ and $\ell_2 = \pi$, then $v(k_1, k_2) = k_1 cos k_2$ is a solution of (4.34), which satisfies (4.35) when $\underset{\alpha(\pm\ell_1,0)}{u}(k_1, k_2) = 2(1-\alpha)k_1 cos k_2$.*

Theorem 4.7.3. *Consider (4.34) and denote $v(k_1 \pm \ell_1, *) = v(k_1+\ell_1, *) + v(k_1-\ell_1, *)$ and $v(*, k_2 \pm \ell_2) = v(*, k_2 + \ell_2) + v(*, k_2 - \ell_2)$. Then, the following four types of numerical solutions of the equation (4.34) are equivalent:*

(a). $$v(k_1, k_2) = (\beta - 2\alpha\gamma)^m v(k_1, k_2 - m\ell_2) + \sum_{r=0}^{m-1} \gamma(\beta - 2\alpha\gamma)^r \left[v(k_1 \pm \ell_1, k_2 - (r+1)\ell_2)\right], \tag{4.37}$$

(b). $$v(k_1, k_2) = \frac{1}{(\beta - 2\alpha\gamma)^m} v(k_1, k_2 + m\ell_2) - \sum_{r=1}^{m} \frac{\gamma}{(\beta - 2\alpha\gamma)^r} v(k_1 \pm \ell_1, k_2 + (r-1)\ell_2), \tag{4.38}$$

(c). $$v(k_1,k_2)=\frac{1}{\gamma^m}v(k_1-m\ell_1,k_2+m\ell_2)-\sum_{r=1}^{m}\frac{\beta-2\alpha\gamma}{\gamma^r}v(k_1-r\ell_1,k_2+(r-1)\ell_2)$$
$$-\sum_{s=0}^{m-1}\frac{1}{\gamma^s}v(k_1-(s+2)\ell_1,k_2+s\ell_2), \tag{4.39}$$

(d). $$v(k_1,k_2)=\frac{1}{\gamma^m}v(k_1+m\ell_1,k_2+m\ell_2)-\sum_{r=1}^{m}\frac{\beta-2\alpha\gamma}{\gamma^r}v(k_1+r\ell_1,k_2+(r-1)\ell_2)$$
$$-\sum_{s=0}^{m-1}\frac{1}{\gamma^s}v(k_1+(s+2)\ell_1,k_2+s\ell_2). \tag{4.40}$$

Proof. (a). From (4.34), we arrive at the relation

$$v(k_1,k_2)=(\beta-2\alpha\gamma)v(k_1,k_2-\ell_2)+\gamma v(k_1\pm\ell_1,k_2-\ell_2). \tag{4.41}$$

By replacing k_2 by $k_2-\ell_2, k_2-2\ell_2, ..., k_m-m\ell_2$ in (4.41), we obtain the expressions for $v(k_1,k_2-r\ell_2)$ and $v(k_1\pm\ell_1,k_2-r\ell_2)$. Now proof of (a) follows from (4.41).

(b). The heat equation (4.34) directly gives the relation

$$v(k_1,k_2)=\frac{1}{(\beta-2\alpha\gamma)}v(k_1,k_2+\ell_2)-\frac{\gamma}{(\beta-2\alpha\gamma)}v(k_1\pm\ell_1,k_2). \tag{4.42}$$

The proof of (b) follows by replacing k_2 by $k_2+\ell_2, k_2+2\ell_2, ..., k_m+m\ell_2$ repeatedly and substituting the corresponding v-values in (4.42).

(c). A simple calculation on (4.34) gives the expression

$v(k_1,k_2)=\dfrac{1}{\gamma}v(k_1-\ell_1,k_2+\ell_2)-\dfrac{\beta-2\alpha\gamma}{\gamma}v(k_1-\ell_1,k_2)-v(k_1-2\ell_1,k_2).$

The proof of (c) follows by replacing k_1 by $k_1-\ell_1, k_1-2\ell_1, ..., k_m-m\ell_1$ and k_2 by $k_2+\ell_2, k_2+2\ell_2, ..., k_m+m\ell_2$ repeatedly.

(d). From $v(k_1,k_2)=\dfrac{1}{\gamma}v(k_1+\ell_1,k_2+\ell_2)-\dfrac{\beta-2\alpha\gamma}{\gamma}v(k_1+\ell_1,k_2)-v(k_1+2\ell_1,k_2),$

the proof of (d) follows by replacing k_1 by $k_1+\ell_1, k_1+2\ell_1, ..., k_m+m\ell_1$ and k_2 by $k_2+\ell_2, k_2+2\ell_2, ..., k_m+m\ell_2$. □

Remark 4.7.4. *Theorem 4.7.3 shows that the present temperature $v(k_1,k_2)$ is obtained by knowing either the past temperature or the future temperature at certain positions. Also, if we know either the past or present temperature with $v(k_1,k_2)$, we can determine the heat transmission factors α, β and γ.*

Example 4.7.5. *The following example shows that the diffusion rate of rod can be identified if the solution $v(k_1,k_2)$ of (4.34) is known and vice versa. Suppose that $v(k_1,k_2)=e^{k_1+k_2}$ is a closed form solution of (4.34), then we have the relation $\underset{\beta(0,\ell_2)}{\Delta} e^{k_1+k_2} = \gamma\big[\underset{\alpha(\ell_1,0)}{\Delta} e^{k_1+k_2} + \underset{\alpha(-\ell_1,0)}{\Delta} e^{k_1+k_2}\big]$, which yields $e^{k_1+k_2+\ell_2}-\beta e^{k_1+k_2}=\gamma\big[e^{k_1+k_2+\ell_1}+e^{k_1+k_2-\ell_1}-2\alpha e^{k_1+k_2}\big]$ and*

$$\gamma = \frac{e^{\ell_2}-\beta}{e^{\ell_1}+e^{-\ell_1}-2\alpha}. \tag{4.43}$$

The MATLAB coding is given below to verify (a) of Theorem 4.7.3, by assuming $m=15$, $k_1=1$, $\ell_1=2$, $k_2=2$, $\ell_2=3$, $\alpha=2$, $\beta=3$, $v(k_1,k_2)=e^{(k_1+k_2)}$.

$exp(3) = (3-4.*((exp(3)-3)./(exp(2)+exp(-2)-4))).\wedge(15).*exp(-42)+ symsum(((exp(3)-3)./(exp(2)+exp(-2)-4)).*(3-4.*((exp(3)-3)./(exp(2)+exp(-2)-4))).\wedge r.*((exp(1+2+2-(r+1).*3))+(exp(1-2+2-(r+1).*3))), r, 0, 14).$

The following theorem gives the condition for Trigonometric function to become a solution of alpha-beta heat equation (4.34).

Theorem 4.7.6. *If* $\underset{\alpha(-\ell_1,0)}{\Delta} v(k_1,k_2) = \delta \underset{\alpha(\ell_1,0)}{\Delta} u(k_1,k_2)$, *then the equation* (4.34) *has a solution* $\cos(k_1+k_2)$ *if and only if either* $\cos(k_1+k_2)=0$ *or* $\sin \ell_1 = 0$.

Proof. From the heat equation (4.34), and the given condition, we arrive

$$\underset{\beta(0,\ell_2)}{\Delta} v(k_1,k_2) = \gamma(1+\delta) \underset{\alpha(\ell_1,0)}{\Delta} v(k_1,k_2). \tag{4.44}$$

If $\cos(k_1+k_2) = \dfrac{[e^{i(k_1+k_2)} + e^{-i(k_1+k_2)}]}{2} = v(k_1,k_2)$, then (4.44) becomes,

$\underset{\beta(0,\ell_2)}{\Delta} \left[e^{i(k_1+k_2)} + e^{-i(k_1+k_2)}\right] = \gamma(1+\delta) \underset{\alpha(\ell_1,0)}{\Delta} \left[e^{i(k_1+k_2)} + e^{-i(k_1+k_2)}\right]$, which yields

$e^{i(k_1+k_2)}\left[e^{il_2} - \beta - \gamma(1+\delta)e^{i\ell_1} - \alpha\right] = e^{-i(k_1+k_2)}\left[e^{i\ell_2} - \beta - \gamma(1+\delta)e^{-i\ell_1} - \alpha\right]$,

which implies either $e^{i(k_1+k_2)} + e^{-i(k_1+k_2)} = 0$ or $e^{i\ell_1} = e^{-i\ell_2}$

and hence $\cos(k_1+k_2) = 0$ or $\sin l_1 = 0$.

The proof of the converse part follows by retracing the above steps. □

4.7.2 Discrete alpha-beta heat equation of thin plate

In the case of thin plate, let $v(k_1,k_2,k_3)$ be the temperature of the plate at position (k_1,k_2) and time k_3. As in the case of rod, the heat equation for the plate is

$$\underset{\beta(0,0,\ell_3)}{\Delta} v(k) = \gamma \underset{\alpha(\pm\ell_{(1,2)})}{\Delta} v(k), \tag{4.45}$$

where $\underset{\alpha(\pm\ell_{(1,2)})}{\Delta} = \underset{\alpha(\ell_1,0,0)}{\Delta} + \underset{\alpha(-\ell_1,0,0)}{\Delta} + \underset{\alpha(0,\ell_2,0)}{\Delta} + \underset{\alpha(0,-\ell_2,0)}{\Delta}$.

Theorem 4.7.7. *Assume that there exists positive integer* m, *and real* $\ell_3 > 0$ *such that* $v(k_1,k_2,k_3-ml_3)$ *and* $\underset{\alpha(\pm\ell_{(1,2)})}{\Delta} v(k_1,k_2,k_3) = \underset{\alpha(\pm\ell_{(1,2)})}{u}(k_1,k_2,k_3)$ *are known* *functions. Then the equation* (4.45) *has a solution as*

$$v(k_1,k_2,k_3) = \beta^m v(k_1,k_2,k_3-m\ell_3) + \gamma\sum_{r=1}^{m}\beta^{r-1}\underset{\alpha(\pm\ell_{(1,2)})}{u}(k_1,k_2,k_3-r\ell_3). \quad (4.46)$$

Proof. The proof is similar to the proof of Theorem 4.7.1. □

Consider the following notations which will be used in the subsequent theorems:

$$v(k_{(1,2)}\pm l_{(1,2)},*) = v(k_1\pm\ell_1,k_2,*) + v(k_1,k_2\pm\ell_2,*),$$

$$v(k_{(2,3)}\pm l_{(2,3)},*) = v(*,k_2\pm\ell_2,k_3) + v(*,k_2,k_3\pm\ell_3),$$

$$v(k_1\pm\ell_1,*,*) = v(k_1+\ell_1,*,*) + v(k_1-\ell_1,*,*),$$

$$v(*,*,k_3\pm\ell_3) = v(*,*,k_3+\ell_3) + v(*,*,k_3-\ell_3).$$

Theorem 4.7.8. *Assume that* $v(k_1,k_2,k_3)$ *is a solution of equation* (4.45)*,* $v(k_1\pm r\ell_1, k_2\pm r\ell_2)$ *exist. Then, the following are equivalent:*

(a). $v(k_1,k_2,k_3) = (\beta-4\alpha\gamma)^m v(k_1,k_2,k_3-m\ell_3) + \sum_{r=0}^{m-1}\gamma(\beta-4\alpha\gamma)^r\times$

$$\big[v(k_1\pm\ell_1,k_2,k_3-(r+1)\ell_3) + v(k_1,k_2\pm\ell_2,k_3-(r+1)\ell_3)\big], \quad (4.47)$$

(b). $v(k_1,k_2,k_3) = \dfrac{1}{(\beta-4\alpha\gamma)^m}v(k_1,k_2,k_3+m\ell_3)$

$$-\sum_{r=1}^{m}\frac{\gamma}{(\beta-4\alpha\gamma)^r}\big[v(k_{(1,2)}\pm l_{(1,2)},k_3+(r-1)\ell_3)\big], \quad (4.48)$$

(c). $v(k) = \dfrac{1}{\gamma^m}v(k_1-m\ell_1,k_2,k_3+m\ell_3) - \sum_{r=1}^{m}\dfrac{\beta-4\alpha\gamma}{\gamma^r}\times$

$$v(k_1-r\ell_1,k_2,k_3+(r-1)\ell_3) - \sum_{r=0}^{m-1}\frac{1}{\gamma^r}v(k_1-(r+2)\ell_1,k_2,k_3+r\ell_3)$$

$$-\sum_{r=0}^{m-1}\frac{1}{\gamma^r}v(k_1-(r+1)\ell_1,k_2\pm\ell_2,k_3+r\ell_3), \quad (4.49)$$

(d). $v(k) = \frac{1}{\gamma^m} v(k_1 + m\ell_1, k_2, k_3 + m\ell_3) - \sum\limits_{r=1}^{m} \frac{\beta - 4\alpha\gamma}{\gamma^r} \times$

$$v(k_1 + r\ell_1, k_2, k_3 + (r-1)\ell_3) - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 + (r+2)\ell_1, k_2, k_3 + r\ell_3)$$

$$- \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 + (r+1)\ell_1, k_2 \pm \ell_2, k_3 + r\ell_3). \tag{4.50}$$

Proof. The proof of this theorem is similar to the proof of the Theorem 4.7.3. □

4.7.3 Discrete alpha-beta heat equation of medium

In most general case, consider homogeneous diffusion medium in $\Re^3$. Let $v(k_1, k_2, k_3, k_4, k_5)$ be the temperature, at position (k_1, k_2, k_3), at time k_4 with density (or pressure) k_5 and denote $k = (k_1, k_2, k_3, k_4, k_5)$. The heat equation for medium by considering pressure and density is formulated as

$$\underset{\beta(\ell_4,\ell_5)}{\Delta} v(k) = \gamma \underset{\alpha(\pm\ell_{(1,2,3)})}{\Delta} v(k), \tag{4.51}$$

where $\underset{\alpha(\pm\ell_{(1,2,3)})}{\Delta} = \underset{\alpha(\ell_1)}{\Delta} + \underset{\alpha(-\ell_1)}{\Delta} + \underset{\alpha(\ell_2)}{\Delta} + \underset{\alpha(-\ell_2)}{\Delta} + \underset{\alpha(\ell_3)}{\Delta} + \underset{\alpha(-\ell_3)}{\Delta}$.

As in the case of rod and thin plate, equation (4.51) has four types of solution as:

(a). $v(k) = (\beta - 6\alpha\gamma)^m v(k_1, k_2, k_3, k_4 - m\ell_4, k_5 - m\ell_5)$

$$+ \sum_{r=0}^{m-1} \gamma(\beta - 6\alpha\gamma)^r \Big[v(k_{(1,2,3)} \pm \ell_{(1,2,3)}, k_4 - (r+1)\ell_4, k_5 - (r+1)\ell_5) \Big], \tag{4.52}$$

(b). $v(k) = \frac{1}{(\beta - 6\alpha\gamma)^m} v(k_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5)$

$$- \sum_{r=1}^{m} \frac{\gamma}{(\beta - 6\alpha\gamma)^r} \Big[v(k_{(1,2,3)} \pm l_{(1,2,3)}, k_4 + (r-1)\ell_4, k_5 + (r-1)\ell_5) \Big], \tag{4.53}$$

$$
\begin{aligned}
\text{(c). } v(k) = {} & \frac{1}{\gamma^m} v(k_1 - m\ell_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5) \\
& - \sum_{r=1}^{m} \frac{\beta - 6\alpha\gamma}{\gamma^r} v(k_1 - r\ell_1, k_2, k_3, k_4 + (r-1)\ell_4, k_5 + (r-1)\ell_5) \\
& - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 - (r+2)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5) \\
& - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 - (r+1)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5), \qquad (4.54)
\end{aligned}
$$

$$
\begin{aligned}
\text{(d). } v(k) = {} & \frac{1}{\gamma^m} v(k_1 + m\ell_1, k_2, k_3, k_4 + m\ell_4, k_5 + m\ell_5) \\
& - \sum_{r=1}^{m} \frac{\beta - 6\alpha\gamma}{\gamma^r} v(k_1 + r\ell_1, k_2, k_3, k_4 + (r-1)\ell_4, k_5 + (r-1)\ell_5) \\
& - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 + (r+2)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5) \\
& - \sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 + (r+1)\ell_1, k_{(2,3)} \pm \ell_{(2,3)}, k_4 + r\ell_4, k_5 + r\ell_5). \qquad (4.55)
\end{aligned}
$$

Thus, we have developed several types of discrete partial difference equation as well as discrete partial q-difference equation with solutions for heat flows in long rod, thin plate and medium. One can obtain Logarithmic functions as exact solutions of these types of partial q-difference equations of heat flows. For usual discrete partial difference equation one can find extorial functions (defined in the seventh chapter) as exact solutions of heat flows.

Chapter 5

Fibonacci Heat Equation Model

5.1 Introduction

In this chapter, partial Fibonacci difference equation is introduced and subjected to investigation in discrete heat equation by having recourse to Fibonacci difference operator with shift values. By having Fourier law of cooling as its basis, the heat transfer in the long rod is investigated by Fibonacci partial difference equation and the solutions obtained are validated by MATLAB.

5.2 Fibonacci difference operator and equation

Consider second order Fibonacci number defined as $F_0 = 1$, $F_1 = x_1$ and $F_n = x_1 F_{n+1} + x_2 F_n$. These second order Fibonacci numbers are used to find solutions of Fibonacci difference equations for heat flows. If the function $v(k_1, k_2)$ is assumed as temperature of a long rod at the position k_1 at time k_2, it will be influenced by certain quantity of heat values at the neighbouring points $k_1 - 2\ell_1$, $k_1 - \ell_1$, $k_1 + \ell_1$, $k_1 + 2\ell_1$ etc. Hence, we obtain the two and three variable heat equation model.

5.2.1 Fibonacci difference operator on two variable

For $x = (x_1, x_2)$, the Fibonacci difference operator on two variable real valued function $v(k)$ with shift values $\ell = (\ell_1, \ell_2)$ and for $k = (k_1, k_2)$ is defined as

$$\underset{x(\ell)}{\Delta} v(k) = v(k) - x_1 v(k - \ell) - x_2 v(k - 2\ell). \tag{5.1}$$

The operator in (5.1) becomes Fibonacci partial difference operator if either ℓ_1 or ℓ_2 is zero but not both. The equations involving a first order linear Fibonacci partial difference equation is given by

$$\underset{x(\ell)}{\Delta} v(k) = u(k), \ell = (0, \ell_2) \quad \text{or} \quad (\ell_1, 0); \quad x = (x_1, x_2) \neq 0. \tag{5.2}$$

The equation (5.2) has a numerical solution of the form

$$v(k_1,k_2) - F_{n+1}v(k_1,k_2-(n+1)\ell_2)$$
$$- x_2F_nv(k_1,k_2-(n+2)\ell_2) = \sum_{i=0}^{n} F_iu(k_1,k_2-i\ell_2). \tag{5.3}$$

5.2.2 Discrete heat equation model with two parameters

Consider the temperature of long rod $v(k_1,k_2)$. By (5.1) and Newton law of cooling, discrete heat equation of rod is expressed as

$$\underset{x(0,\ell_2)}{\Delta} v(k_1,k_2) = \gamma \underset{x(\pm\ell_1,0)}{\Delta} v(k_1,k_2); \quad x=(x_1,x_2), \tag{5.4}$$

where $\underset{x(\pm\ell_1,0)}{\Delta} = \underset{x(\ell_1,0)}{\Delta} + \underset{x(-\ell_1,0)}{\Delta}$.
Our main aim is to study and discuss the solution of the Fibonacci partial difference equation (5.4). Here, we derive the temperature formula for $v(k_1,k_2)$ at the position k_1 and at time k_2.

Theorem 5.2.1. *If* $\underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2) = \underset{x(\pm\ell_1)}{u}(k_1,k_2)$ *are known, then the heat equation* (5.4) *has a solution*

$$v(k_1,k_2) = F_{n+1}v(k_1,k_2-(n+1)\ell_2)$$
$$+ x_2F_nv(k_1,k_2-(n+2)\ell_2) + \gamma\sum_{i=0}^{n} F_i \underset{x(\pm\ell_1)}{u}(k_1,k_2-i\ell_2). \tag{5.5}$$

Proof. By representing $\underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2) = \underset{x(\pm\ell_1)}{u}(k_1,k_2)$, (5.4) becomes

$$v(k_1,k_2) = F_{n+1}v(k_1,k_2-(n+1)\ell_2)$$

$$+x_2F_nv(k_1,k_2-(n+2)\ell_2)+\gamma \overset{-1}{\underset{x(0,\ell_2)}{\Delta}} \underset{x(\pm\ell_1)}{u}(k_1,k_2). \tag{5.6}$$

The proof of (5.5) follows from the relation,

$$\overset{-1}{\underset{x(0,\ell_2)}{\Delta}} \underset{x(\pm\ell_1)}{u}(k_1,k_2) = \sum_{i=1}^{n} F_i \underset{x(\pm\ell_1)}{u}(k_1-r(0),k_2-i\ell_2) \text{ in } (5.6). \qquad \square$$

Theorem 5.2.2. *Consider (5.4) and denote $v(k_1\pm\ell_1,*) = v(k_1+\ell_1,*)+v(k_1-\ell_1,*)$ and $v(k_1\pm 2\ell_1,*) = v(k_1+2\ell_1,*)+v(k_1-2\ell_1,*)$. Then, the following four types solutions of the equation (5.4) are equivalent:*

(a). $$v(k_1,k_2) = \frac{{x_1}^m}{(1-2\gamma)^m}v(k_1,k_2-m\ell_2) - \sum_{i=1}^{m}\frac{\gamma {x_1}^i}{(1-2\gamma)^i}v(k_1\pm\ell_1,k_2-(i-1)\ell_2)$$

$$+\sum_{i=1}^{m}\frac{x_2{x_1}^{i-1}}{(1-2\gamma)^i}\Big[v(k_1,k_2-(i+1)\ell_2)-\gamma v(k_1\pm 2\ell_1,k_2-(i-1)\ell_2)\Big], \tag{5.7}$$

(b). $$v(k_1,k_2) = \frac{(1-2\gamma)^m}{{x_1}^m}v(k_1,k_2+m\ell_2) + \sum_{i=1}^{m}\frac{\gamma(1-2\gamma)^{i-1}}{{x_1}^{i-1}}v(k_1\pm\ell_1,k_2+i\ell_2)$$

$$-\sum_{i=1}^{m}\frac{x_2(1-2\gamma)^{i-1}}{{x_1}^i}\Big[v(k_1,k_2+(i-2)\ell_2)-\gamma v(k_1\pm 2\ell_1,k_2+i\ell_2)\Big], \tag{5.8}$$

(c). $$v(k_1,k_2) = \frac{1}{\gamma^m}v(k_1-m\ell_1,k_2-m\ell_2)$$

$$-\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^{i-1}}\Big[v(k_1-(i-2)\ell_1,k_2-(i-1)\ell_2)+v(k_1-(i+2)\ell_1,k_2-(i-1)\ell_2)\Big]$$

$$+\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^i}v(k_1-i\ell_1,k_2-(i+1)\ell_2)-\sum_{i=0}^{m}\frac{1}{\gamma^{i-1}}v(k_1-(i+1)\ell_1,k_2-(i-1)\ell_2)$$

$$-\sum_{i=0}^{m}\frac{(1-2\gamma)}{x_1\gamma^i}v(k_1-i\ell_1,k_2-(i-1)\ell_2), \tag{5.9}$$

(d). $$v(k_1,k_2) = \frac{1}{\gamma^m}v(k_1+m\ell_1,k_2-m\ell_2)$$

$$-\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^{i-1}}\Big[v(k_1+(i+2)\ell_1,k_2-(i-1)\ell_2)+v(k_1+(i-2)\ell_1,k_2-(i-1)\ell_2)\Big]$$
$$+\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^{i}}v(k_1+i\ell_1,k_2-(i+1)\ell_2)-\sum_{i=0}^{m}\frac{1}{\gamma^{i-1}}v(k_1+(i+1)\ell_1,k_2-(i-1)\ell_2)$$
$$-\sum_{i=0}^{m}\frac{(1-2\gamma)}{x_1\gamma^{i}}v(k_1+i\ell_1,k_2-(i-1)\ell_2). \tag{5.10}$$

Proof. (a). By applying the difference operator $\underset{x(\ell)}{\Delta}$ given in (5.1) on (5.4),

$$v(k_1,k_2)=\frac{x_1}{(1-2\gamma)}v(k_1,k_2-\ell_2)+\frac{x_2}{(1-2\gamma)}v(k_1,k_2-2\ell_2)$$
$$-\frac{x_1\gamma}{(1-2\gamma)}v(k_1\pm\ell_1,k_2)-\frac{x_2\gamma}{(1-2\gamma)}v(k_1\pm2\ell_1,k_2). \tag{5.11}$$

Replacing k_2 by $k_2-\ell_2,k_2-2\ell_2,...,k_2-m\ell_2$ in (5.11), $v(k_1,k_2-r\ell_2)$ and $v(k_1\pm\ell_1,k_2-r\ell_2)$. Now proof of (a) follows by applying the values in (5.11).

(b). By expanding the operator in the heat equation (5.4), we arrive at

$$v(k_1,k_2)=\frac{(1-2\gamma)}{x_1}v(k_1,k_2+\ell_2)+\gamma v(k_1\pm\ell_1,k_2+\ell_2)$$
$$-\frac{x_2}{x_1}\Big[v(k_1,k_2-\ell_2)-\gamma v(k_1\pm2\ell_1,k_2+\ell_2)\Big]. \tag{5.12}$$

The proof of (b) follows by replacing k_2 by $k_2+\ell_2,k_2+2\ell_2,...,k_2+m\ell_2$ repeatedly and substituting corresponding γ-values in (5.12).

(c). A simple calculation on (5.4) gives the expression

$$v(k_1,k_2)=\frac{1}{\gamma}v(k_1-\ell_1,k_2-\ell_2)+\frac{x_2}{x_1\gamma}v(k_1-\ell_1,k_2-2\ell_2)-v(k_1-2\ell_1,k_2)$$
$$-\frac{x_2}{x_1}\Big[v(k_1+\ell_1,k_2)+v(k_1-3\ell_1,k_2)\Big]-\frac{(1-2\gamma)}{x_1\gamma}v(k_1-\ell_1,k_2)$$

The proof of (c) follows by replacing k_1 by $k_1-\ell_1,k_1-2\ell_1,...,k_1-m\ell_1$ and k_2 by $k_2-\ell_2,k_2-2\ell_2,...,k_2-m\ell_2$ repeatedly.

(d). The expansion of (5.4) gives the expression

$$v(k_1,k_2) = \frac{1}{\gamma}v(k_1+\ell_1,k_2-\ell_2) + \frac{x_2}{x_1\gamma}v(k_1+\ell_1,k_2-2\ell_2) - v(k_1+2\ell_1,k_2)$$
$$-\frac{x_2}{x_1}\Big[v(k_1+3\ell_1,k_2)+v(k_1-\ell_1,k_2)\Big] - \frac{(1-2\gamma)}{x_1\gamma}v(k_1+\ell_1,k_2).$$

The proof of (d) follows by replacing k_1 by $k_1+\ell_1, k_1+2\ell_1, ..., k_1+m\ell_1$ and k_2 by $k_2-\ell_2, k_2-2\ell_2, ..., k_2-m\ell_2$ repeatedly. Fibonacci heat equation models can be applied for getting more accuracy in heat transfer. □

The following example is an illustration of Fibonacci partial heat equation (5.4). Here, we discuss the exponential solution of heat equation (5.1).

Example 5.2.3. *The dissemination rate of rod is identified by the given example if the solution $v(k_1,k_2)$ of (5.4) is known. Suppose that $v(k_1,k_2) = e^{k_1+k_2}$ is an exact solution of (5.4), then we have the relation*

$\underset{0,\ell_2(x)}{\Delta} e^{k_1+k_2} = \gamma\big[\underset{\ell_1(x)}{\Delta} e^{k_1+k_2} + \underset{-\ell_1(x)}{\Delta} e^{k_1+k_2}\big]$, *which yields*

$e^{k_1+k_2} - x_1e^{k_1+k_2-\ell_2} - x_2e^{k_1+k_2-2\ell_2} = \gamma\big[e^{k_1+k_2} - x_1e^{k_1\pm\ell_1+k_2} - x_2e^{k_1\pm2\ell_1+k_2}\big]$.

Cancelling $e^{k_1+k_2}$ on both sides derives

$$\gamma = \frac{1-x_1e^{-\ell_2}-x_2e^{-2\ell_2}}{2-x_1(e^{\ell_1}+e^{-\ell_1})-x_2(e^{2\ell_1}+e^{-2\ell_1})}. \tag{5.13}$$

For numerical verification, we give the MATLAB coding for (a) of Theorem 5.2.2. When $m=1$, $k_1=1$, $\ell_1=1$, $k_2=2$, $\ell_2=2$, $x_1=1$, $x_2=2$, $v(k_1,k_2)=e^{(k_1+k_2)}$ and γ is as given in (5.13), we have the following coding

$((1.\wedge 1)./(1.102638526.\wedge 1)).*exp(3-(1.*2))-(symsum((((-0.051319263).*(1.\wedge i))./(1.102638526.\wedge (i))).*(exp(4-((i-1).*2))+(exp(2-((i-1).*$

$2)))), i, 1, 1)) + (symsum(((2. * (1. \wedge (i-1)))./(1.102638526. \wedge i)). * (exp(3-((i+1). * 2)) + (0.051319263. * ((exp(5-((i-1). * 2)) + exp(1-((i-1). * 2)))))), i, 1, 1))$.

5.2.3 Fibonacci heat equation model with three parameters

For getting accuracy value of heat transmission, (5.1) can be replaced by Fibonacci partial difference operator with three parameter $x = (x_1, x_2, x_3)$ as

$$\underset{x(0,\ell_2)}{\Delta} v(k_1, k_2) = v(k_1, k_2) - x_1 v(k_1, k_2 - \ell_2) \\ - x_2 v(k_1, k_2 - 2\ell_2) - x_3 v(k_1, k_2 - 3\ell_2). \tag{5.14}$$

In this case, the corresponding heat equation model is taken as

$$\underset{x(0,\ell_2)}{\Delta} v(k_1, k_2) = \gamma \underset{x(\pm\ell_1)}{\Delta} v(k_1, k_2); \quad x = (x_1, x_2, x_3). \tag{5.15}$$

As in the proof of Theorem 5.2.1, we get the following theorem.

Theorem 5.2.4. *Assume that there exists a positive integer n, and a real number $\ell_2 > 0$ such that $v(k_1, k_2 - n\ell_2)$ and $\underset{x(\pm\ell_1)}{\Delta} v(k_1, k_2) = \underset{x(\pm\ell_1)}{u}(k_1, k_2)$ are known. Then, the heat equation (5.15) has a solution $v(k_1, k_2)$ of the form*

$$v(k_1, k_2) = F_{n+1} v(k_1, k_2 - (n+1)\ell_2) + (x_2 F_n + x_3 F_{n-1}) v(k_1, k_2 - (n+2)\ell_2) \\ + x_3 F_n v(k_1, k_2 - (n+3)\ell_2) + \gamma \sum_{i=0}^{n} F_i \underset{\pm\ell_1(x)}{u}(k_1, k_2 - i\ell_2), \tag{5.16}$$

where $F_0 = 1$, $F_1 = x_1$, $F_2 = x_1 F_1 + x_2 F_0$ and $F_{n+3} = x_1 F_{n+2} + x_2 F_{n+1} + x_3 F_n$, $n \geq 0$.

We use the following notations in the subsequent theorems:

$v(k_1 \pm \ell_1, *) = v(k_1+\ell_1, *)+v(k_1-\ell_1, *)$, $v(k_1 \pm 2\ell_1, *) = v(k_1+2\ell_1, *)+v(k_1-2\ell_1, *)$,

$v(k_1 \pm 3\ell_1, *) = v(k_1+3\ell_1, *)+v(k_1-3\ell_1, *)$.

The following theorem is an improvement of Theorem 5.2.2 of heat transmission.

Theorem 5.2.5. *Consider the equation* (5.15). *Then, the following four types solutions of the equation* (5.15) *are equivalent:*

(a).
$$v(k_1,k_2) = \frac{{x_1}^m}{(1-2\gamma)^m} v(k_1, k_2 - m\ell_2) - \sum_{i=1}^{m} \frac{\gamma {x_1}^i}{(1-2\gamma)^i} v(k_1 \pm \ell_1, k_2-(i-1)\ell_2)$$
$$+ \sum_{i=1}^{m} \frac{x_2 {x_1}^{i-1}}{(1-2\gamma)^i} \Big[v(k_1, k_2-(i+1)\ell_2) - \gamma v(k_1 \pm 2\ell_1, k_2-(i-1)\ell_2) \Big]$$
$$+ \sum_{i=1}^{m} \frac{x_3 {x_1}^{i-1}}{(1-2\gamma)^i} \Big[v(k_1, k_2-(i+2)\ell_2) - \gamma v(k_1 \pm 3\ell_1, k_2-(i-1)\ell_2) \Big], \tag{5.17}$$

(b).
$$v(k_1,k_2) = \frac{(1-2\gamma)^m}{{x_1}^m} v(k_1, k_2 + m\ell_2) + \sum_{i=1}^{m} \frac{\gamma(1-2\gamma)^{i-1}}{{x_1}^{i-1}} v(k_1 \pm \ell_1, k_2+i\ell_2)$$
$$- \sum_{i=1}^{m} \frac{x_2(1-2\gamma)^{i-1}}{{x_1}^i} \Big[v(k_1, k_2+(i-2)\ell_2) - \gamma v(k_1 \pm 2\ell_1, k_2+i\ell_2) \Big]$$
$$- \sum_{i=1}^{m} \frac{x_3(1-2\gamma)^{i-1}}{{x_1}^i} \Big[v(k_1, k_2+(i-3)\ell_2) - \gamma v(k_1 \pm 3\ell_1, k_2+i\ell_2) \Big], \tag{5.18}$$

(c).
$$v(k_1,k_2) = \frac{1}{\gamma^m} v(k_1 - m\ell_1, k_2 - m\ell_2) + \sum_{r=1}^{m} \frac{x_2}{x_1\gamma^i} v(k_1 - i\ell_1, k_2-(i+1)\ell_2)$$
$$+ \sum_{r=1}^{m} \frac{x_3}{x_1\gamma^i} v(k_1 - i\ell_1, k_2-(i+2)\ell_2) - \sum_{i=0}^{m} \frac{1}{\gamma^{i-1}} v(k_1-(i+1)\ell_1, k_2-(i-1)\ell_2)$$
$$- \sum_{r=1}^{m} \frac{x_2}{x_1\gamma^{i-1}} \Big[v(k_1-(i-2)\ell_1, k_2-(i-1)\ell_2) + v(k_1-(i+2)\ell_1, k_2-(i-1)\ell_2) \Big]$$
$$- \sum_{r=1}^{m} \frac{x_3}{x_1\gamma^{i-1}} \Big[v(k_1-(i-3)\ell_1, k_2-(i-1)\ell_2) + v(k_1-(i+3)\ell_1, k_2-(i-1)\ell_2) \Big]$$
$$- \sum_{i=0}^{m} \frac{(1-2\gamma)}{x_1\gamma^i} v(k_1 - i\ell_1, k_2-(i-1)\ell_2), \tag{5.19}$$

(d).
$$v(k_1,k_2)=\frac{1}{\gamma^m}v(k_1+m\ell_1,k_2-m\ell_2)+\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^i}v(k_1+i\ell_1,k_2-(i+1)\ell_2)$$
$$+\sum_{r=1}^{m}\frac{x_3}{x_1\gamma^i}v(k_1+i\ell_1,k_2-(i+2)\ell_2)-\sum_{i=0}^{m}\frac{1}{\gamma^{i-1}}v(k_1+(i+1)\ell_1,k_2-(i-1)\ell_2)$$
$$-\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^{i-1}}\Big[v(k_1+(i+2)\ell_1,k_2-(i-1)\ell_2)+v(k_1+(i-2)\ell_1,k_2-(i-1)\ell_2)\Big]$$
$$-\sum_{r=1}^{m}\frac{x_3}{x_1\gamma^{i-1}}\Big[v(k_1+(i+3)\ell_1,k_2-(i-1)\ell_2)+v(k_1+(i-3)\ell_1,k_2-(i-1)\ell_2)\Big]$$
$$-\sum_{i=0}^{m}\frac{(1-2\gamma)}{x_1\gamma^i}v(k_1+i\ell_1,k_2-(i-1)\ell_2). \tag{5.20}$$

Proof. The proof of this theorem is similar to the proof of Theorem 5.2.2. □

Example 5.2.6. *The following example shows that the diffusion rate of rod can be identified if the solution $v(k_1,k_2)$ is known and vice versa. Suppose that $v(k_1,k_2)=e^{k_1+k_2}$ is a closed form solution of (5.15), then we have the relation*
$$\underset{0,\ell_2(x)}{\Delta}e^{k_1+k_2}=\gamma\big[\underset{\ell_1(x)}{\Delta}e^{k_1+k_2}+\underset{-\ell_1(x)}{\Delta}e^{k_1+k_2}\big],$$
which yields $e^{k_1+k_2}-x_1e^{k_1+k_2-\ell_2}-x_2e^{k_1+k_2-2\ell_2}-x_3e^{k_1+k_2-3\ell_2}$
$$=\gamma\big[e^{k_1+k_2}-x_1e^{k_1\pm\ell_1+k_2}-x_2e^{k_1\pm2\ell_1+k_2}-x_3e^{k_1\pm3\ell_1+k_2}\big].$$
Cancelling $e^{k_1+k_2}$ on both sides we get
$$\gamma=\frac{1-x_1e^{-\ell_2}-x_2e^{-2\ell_2}-x_3e^{-3\ell_2}}{2-x_1(e^{\ell_1}+e^{-\ell_1})-x_2(e^{2\ell_1}+e^{-2\ell_1})-x_3(e^{3\ell_1}+e^{-3\ell_1})}. \tag{5.21}$$
For numerical verification the MATLAB coding for (a), (b), (c) and (d) are as similar as that of two parameters given the previous section.

5.3 Formation of Fibonacci heat equation

Here, we extend the theories in sections 5.2.2 and 5.2.3. For $x = (x_1, x_2, x_3, ..., x_r)$, the x-difference operator with r-parameters on real valued function $v(k), k = (k_1, k_2)$ with shift values $\ell = (\ell_1, \ell_2)$ is defined as

$$\underset{x(\ell)}{\Delta} v(k_1, k_2) = v(k_1, k_2) - x_1 v(k_1 - \ell_1, k_2 - \ell_2) - x_2 v(k_1 - 2\ell_1, k_2 - 2\ell_2)$$

$$- x_3 v(k_1 - 3\ell_1, k_2 - 3\ell_2) - .... - x_r v(k_1 - r\ell_1, k_2 - r\ell_2). \quad (5.22)$$

The corresponding generalized Fibonacci partial difference equation with two parameters is given by

$$\underset{x(\ell)}{\Delta} v(k) = u(k), \ell = (0, \ell_2) \quad \text{or} \quad (\ell_1, 0); \quad x = (x_1, x_2, ..., x_r), \quad (5.23)$$

using inverse principle, the equation (5.23) has a numerical solution of the form

$$v(k_1, k_2) - \sum_{i=0}^{m} \sum_{j=i}^{m} x_j F_{n+i-j} v(k_1, k_2 - (n+i)\ell_2) = \sum_{i=0}^{n} F_i u(k_1, k_2 - i\ell_2), \quad (5.24)$$

where $F_n = 0$ when $n < 0$, $F_0 = 1$, $F_1 = x_1$, $F_2 = x_1 F_1 + x_2 F_0$,

$F_3 = x_1 F_2 + x_1 F_1 + x_3 F_0$, and $F_n = x_1 F_{n-1} + x_2 F_{n-2} + ... + x_r F_{n-r}$ for $n > r$.

5.3.1 Generalized Fibonacci heat equation of long rod

Here, we use the operator given in (5.22) to form this model. Let $v(k_1, k_2)$ be the temperature at the position k_1 and time k_2 of long rod [9]. By (5.22) and

Newton law of cooling, discrete heat equation of rod is expressed as

$$\underset{x(0,\ell_2)}{\Delta} v(k_1,k_2) = \gamma \underset{x(\pm\ell_1,0)}{\Delta} v(k_1,k_2); \quad x = (x_1, x_2, ..., x_r), \tag{5.25}$$

where $\underset{x(\pm\ell_1,0)}{\Delta} = \underset{x(\ell_1,0)}{\Delta} + \underset{x(-\ell_1,0)}{\Delta}$. Here, we derive the temperature formula for $v(k_1,k_2)$ at the common position (k_1,k_2).

Theorem 5.3.1. *If* $\underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2) = \underset{x(\pm\ell_1)}{u}(k_1,k_2)$ *are known, then the heat equation* (5.25) *has a solution of the form*

$$v(k_1,k_2) = \sum_{i=0}^{m}\sum_{j=i}^{m} x_j F_{n+i-j} v(k_1, k_2-(n+i)\ell_2) + \gamma \sum_{i=0}^{n} F_i \underset{x(\pm\ell_1)}{u}(k_1, k_2 - i\ell_2). \tag{5.26}$$

Proof. By representing $\underset{x(\pm\ell_1)}{\Delta} v(k_1,k_2) = \underset{x(\pm\ell_1)}{u}(k_1,k_2)$, (5.25) becomes

$$v(k_1,k_2) = \sum_{i=0}^{m}\sum_{j=i}^{m} x_j F_{n+i-j} v(k_1, k_2-(n+i)\ell_2) + \gamma \overset{-1}{\underset{x(0,\ell_2)}{\Delta}} \underset{x(\pm\ell_1)}{u}(k_1,k_2). \tag{5.27}$$

The proof of (5.26) follows from the relation,

$$\overset{-1}{\underset{x(0,\ell_2)}{\Delta}} \underset{x(\pm\ell_1)}{u}(k_1,k_2) = \sum_{i=1}^{n} F_i \underset{x(\pm\ell_1)}{u}(k_1 - r(0), k_2 - i\ell_2) \text{ and } (5.27). \qquad \square$$

Theorem 5.3.2. *Consider* (5.25) *and denote* $v(k_1 \pm \ell_1, *) = v(k_1+\ell_1, *) + v(k_1-\ell_1, *)$ *and* $v(k_1 \pm 2\ell_1, *) = v(k_1+2\ell_1, *) + v(k_1-2\ell_1, *)$*. Then, the following four types of solutions of the equation* (5.25) *are equivalent:*

(a).
$$v(k_1,k_2) = \frac{{x_1}^n}{(1-2\gamma)^n} v(k_1, k_2 - n\ell_2) - \sum_{i=1}^{n} \frac{\gamma {x_1}^i}{(1-2\gamma)^i} v(k_1 \pm \ell_1, k_2 - (i-1)\ell_2)$$
$$+ \sum_{r=2}^{n} \left\{ \sum_{i=1}^{r} \frac{x_r {x_1}^{i-1}}{(1-2\gamma)^i} \Big[v(k_1, k_2 - (i+(r-1))\ell_2) - \gamma v(k_1 \pm r\ell_1, k_2 - (i-1)\ell_2) \Big] \right\}, \tag{5.28}$$

(b). $$v(k_1,k_2)=\frac{(1-2\gamma)^n}{{x_1}^n}v(k_1,k_2+n\ell_2)+\sum_{i=1}^{n}\frac{\gamma(1-2\gamma)^{i-1}}{{x_1}^{i-1}}v(k_1\pm\ell_1,k_2+i\ell_2)$$

$$\sum_{r=2}^{n}\left\{\sum_{i=1}^{r}\frac{x_r(1-2\gamma)^{i-1}}{{x_1}^{i}}\Big[v(k_1,k_2+(i-r)\ell_2)\ -\gamma v(k_1\pm r\ell_1,k_2+i\ell_2)\Big]\right\}, \quad (5.29)$$

(c). $$v(k_1,k_2)=\frac{1}{\gamma^n}v(k_1-n\ell_1,k_2-n\ell_2)-\sum_{i=0}^{m}\frac{1}{\gamma^{i-1}}v(k_1-(i+1)\ell_1,k_2-(i-1)\ell_2)$$

$$-\sum_{i=0}^{m}\frac{(1-2\gamma)}{x_1\gamma^i}v(k_1-i\ell_1,k_2-(i-1)\ell_2)$$

$$-\sum_{r=2}^{n}\left\{\sum_{i=1}^{r}\frac{x_r}{x_1\gamma^{i-1}}\Big[v(k_1-(i-r)\ell_1,k_2-(i-1)\ell_2)\ +v(k_1-(i+r)\ell_1,k_2-(i-1)\ell_2)\Big]\right\}$$

$$+\sum_{r=2}^{n}\left\{\sum_{i=1}^{r}\frac{x_r}{x_1\gamma^i}v(k_1-i\ell_1,k_2-(i+(r-1))\ell_2)\right\}, \quad (5.30)$$

(d). $$v(k_1,k_2)=\frac{1}{\gamma^n}v(k_1+n\ell_1,k_2-n\ell_2)-\sum_{i=0}^{m}\frac{1}{\gamma^{i-1}}v(k_1+(i+1)\ell_1,k_2-(i-1)\ell_2)$$

$$-\sum_{i=0}^{m}\frac{(1-2\gamma)}{x_1\gamma^i}v(k_1+i\ell_1,k_2-(i-1)\ell_2)$$

$$-\sum_{r=2}^{n}\left\{\sum_{i=1}^{r}\frac{x_r}{x_1\gamma^{i-1}}\Big[v(k_1+(i-r)\ell_1,k_2-(i-1)\ell_2)\ +v(k_1+(i+r)\ell_1,k_2-(i-1)\ell_2)\Big]\right\}$$

$$+\sum_{r=2}^{n}\left\{\sum_{i=1}^{r}\frac{x_r}{x_1\gamma^i}v(k_1+i\ell_1,k_2-(i+(r-1))\ell_2)\right\}. \quad (5.31)$$

Proof. (a). From (5.25), we arrive at the relation

$$v(k_1,k_2)=\frac{x_1}{(1-2\gamma)}v(k_1,k_2-\ell_2)+...+\frac{x_r}{(1-2\gamma)}v(k_1,k_2-r\ell_2)$$
$$-\frac{x_1\gamma}{(1-2\gamma)}v(k_1\pm\ell_1,k_2)-...-\frac{x_r\gamma}{(1-2\gamma)}v(k_1\pm r\ell_1,k_2). \quad (5.32)$$

By replacing k_2 by $k_2-\ell_2,k_2-2\ell_2,...,k_2-m\ell_2$ in (5.32), we get the expressions for $v(k_1,k_2-r\ell_2)$ and $v(k_1\pm r\ell_1,k_2)$. Now the proof of (a) follows by applying all these values in (5.32).

(b). The heat equation (5.25) gives the relation

$$
\begin{aligned}
v(k_1,k_2) = {} & \frac{(1-2\gamma)}{x_1} v(k_1,k_2+\ell_2) + \gamma v(k_1 \pm \ell_1, k_2+\ell_2) \\
& - \frac{x_2}{x_1}\Big[v(k_1,k_2-\ell_2) - \gamma v(k_1 \pm 2\ell_1, k_2+\ell_2) \Big] \\
& - \dots - \frac{x_{r-1}}{x_1}\Big[v(k_1,k_2-\ell_2) - \gamma v(k_1 \pm (r-1)\ell_1, k_2+\ell_2) \Big] \\
& - \frac{x_r}{x_1}\Big[v(k_1,k_2-\ell_2) - \gamma v(k_1 \pm r\ell_1, k_2+\ell_2) \Big]. \qquad (5.33)
\end{aligned}
$$

The proof of (b) follows by replacing k_2 by $k_2+\ell_2, k_2+2\ell_2, \dots, k_2+m\ell_2$ repeatedly and substituting corresponding γ-values in (5.33).

(c). A simple calculation on (5.25) gives the expression

$$
\begin{aligned}
v(k_1,k_2) = {} & \frac{1}{\gamma} v(k_1-\ell_1,k_2-\ell_2) - v(k_1-2\ell_1,k_2) + \frac{x_2}{x_1\gamma} v(k_1-\ell_1,k_2-2\ell_2) \\
& + \frac{x_3}{x_1\gamma} v(k_1-\ell_1,k_2-3\ell_2) + \dots + \frac{x_r}{x_1\gamma} v(k_1-\ell_1,k_2-r\ell_2) \\
& - \frac{x_2}{x_1}\Big[v(k_1+\ell_1,k_2) + v(k_1-3\ell_1,k_2) \Big] - \frac{x_3}{x_1}\Big[v(k_1+2\ell_1,k_2) + v(k_1-4\ell_1,k_2) \Big] \\
& - \dots - \frac{x_r}{x_1}\Big[v(k_1+(r-1)\ell_1,k_2) + v(k_1-(r+1)\ell_1,k_2) \Big] - \frac{(1-2\gamma)}{x_1\gamma} v(k_1-\ell_1,k_2).
\end{aligned}
$$

The proof of (c) follows by replacing k_1 by $k_1-\ell_1, k_1-2\ell_1, \dots, k_1-m\ell_1$ and k_2 by $k_2-\ell_2, k_2-2\ell_2, \dots, k_2-m\ell_2$ repeatedly.

(d). By expanding the equation (5.25), we get the expression

$$
\begin{aligned}
v(k_1,k_2) = {} & \frac{1}{\gamma} v(k_1+\ell_1,k_2-\ell_2) - v(k_1+2\ell_1,k_2) + \frac{x_2}{x_1\gamma} v(k_1+\ell_1,k_2-2\ell_2) \\
& + \frac{x_3}{x_1\gamma} v(k_1+\ell_1,k_2-3\ell_2) + \dots + \frac{x_r}{x_1\gamma} v(k_1+\ell_1,k_2-r\ell_2) \\
& - \frac{x_2}{x_1}\Big[v(k_1+3\ell_1,k_2) + v(k_1-\ell_1,k_2) \Big] - \frac{x_3}{x_1}\Big[v(k_1+4\ell_1,k_2) + v(k_1-2\ell_1,k_2) \Big] \\
& - \dots - \frac{x_r}{x_1}\Big[v(k_1+(r+1)\ell_1,k_2) + v(k_1-(r_1)\ell_1,k_2) \Big] - \frac{(1-2\gamma)}{x_1\gamma} v(k_1+\ell_1,k_2).
\end{aligned}
$$

The proof of (d) follows by replacing k_1 by $k_1+\ell_1, k_1+2\ell_1, \dots, k_1+m\ell_1$ and k_2 by

$k_2 - \ell_2, k_2 - 2\ell_2, ..., k_2 - m\ell_2$ repeatedly. $\square$

The following example is the numerical verification of Fibonacci heat equation of rod by exponential solutions.

Example 5.3.3. *The diffusion rate of rod can be identified if the solution $v(k_1, k_2)$ of (5.25) is known and vice versa. Suppose that $v(k_1, k_2) = e^{k_1+k_2}$ is a closed form solution of (5.25), then we have the relation*

$$\underset{0,\ell_2(x)}{\Delta} e^{k_1+k_2} = \gamma\big[\underset{\ell_1(x)}{\Delta} e^{k_1+k_2} + \underset{-\ell_1(x)}{\Delta} e^{k_1+k_2}\big],$$

which yields $e^{k_1+k_2} - x_1 e^{k_1+k_2-\ell_2} - ... - x_r e^{k_1+k_2-r\ell_2} = \gamma\big[e^{k_1+k_2} - x_1 e^{k_1 \pm \ell_1 + k_2} - ... - x_r e^{k_1 \pm r\ell_1 + k_2}\big]$.

Cancelling $e^{k_1+k_2}$ on both sides we derive

$$\gamma = \frac{1 - x_1 e^{-\ell_2} - x_2 e^{-2\ell_2}}{2 - x_1(e^{\ell_1} + e^{-\ell_1}) - x_2(e^{2\ell_1} + e^{-2\ell_1})}. \tag{5.34}$$

We give the MATLAB coding for (a) of Theorem 5.2.2 when $m = 1$, $r = 2$, $k_1 = 1$, $\ell_1 = 1$, $k_2 = 2$, $\ell_2 = 2$, $x_1 = 1$, $x_2 = 2$, $v(k_1, k_2) = e^{(k_1+k_2)}$ and γ is as given in (5.34). The code is as follows:

$((1. \wedge 1)./(1.102638526. \wedge 1)). * exp(3 - (1. * 2)) - (symsum((((-0.051319263). * (1. \wedge i))./(1.102638526. \wedge (i))). * (exp(4 - ((i - 1). * 2)) + (exp(2 - ((i - 1). * 2)))), i, 1, 1)) + (symsum(((2. * (1. \wedge (i - 1)))./(1.102638526. \wedge i)). * (exp(3 - ((i + 1). * 2)) + (0.051319263. * ((exp(5 - ((i - 1). * 2)) + exp(1 - ((i - 1). * 2)))))), i, 1, 1)).$

5.4 Fibonacci delay heat equation models

In this section, we extend theory of discrete heat equation, discrete partial heat equation, discrete α-β heat equation and discrete q-heat equations to discrete Fibonacci heat equation for long rod, thin plate and medium for getting more accuracy results.

Also, the higher order partial difference equation of heat flow of thin plate is

$$\underset{x(0,0,\ell_3)}{\Delta} v(k) = \gamma\Big[\underset{x(\ell_1,\ell_2,0)}{\Delta} v(k) + \underset{x(-\ell_1,-\ell_2,0)}{\Delta} v(k)\Big], \tag{5.35}$$

where $\underset{x(\ell)}{\Delta} v(k) = v(k) - x_1 v(k-\ell) - x_2 v(k-2\ell) - ... - x_n v(k-n\ell)$ if $x = (x_1, x_2, .., x_n)$ and $\ell = (\ell_1, \ell_2, \ell_3)$.

5.4.1 Fibonacci delay heat equation of long rod

Consider a long rod with $v(k_1, k_2)$ as temperature, in which k_1 and k_2 denote position and time respectively [9]. By Fourier's cooling law and using (5.1), the discrete delay heat equation is obtained as

$$\underset{x(0,\ell_2)}{\Delta} v(k_1, k_2) = \gamma \underset{x(\pm\ell_1,0)}{\Delta} v(k_1, k_2 - \sigma); \quad x = (x_1, x_2, ..., x_r), \tag{5.36}$$

where σ is a delay factor and $\underset{x(\pm\ell_1,0)}{\Delta} = \underset{x(\ell_1,0)}{\Delta} + \underset{x(-\ell_1,0)}{\Delta}$. The objective of this section is to study and discuss the solution of the heat equation (5.36) with Fibonacci operator of r^{th} order.

Theorem 5.4.1. *If* $\underset{x(\pm\ell_1)}{\Delta} v(k_1, k_2-\sigma) = \underset{x(\pm\ell_1)}{u}(k_1, k_2-\sigma)$ *is given then the delay heat equation has a solution of the form*

$$v(k_1,k_2) = \sum_{i=0}^{m}\sum_{j=i}^{m} x_j F_{n+i-j} v(k_1, k_2-(n+i)\ell_2) + \gamma \sum_{i=0}^{n} F_i \underset{x(\pm\ell_1)}{u}(k_1, k_2-i\ell_2-\sigma). \tag{5.37}$$

Proof. By representing $\underset{x(\pm\ell_1)}{\Delta} v(k_1, k_2-\sigma) = \underset{x(\pm\ell_1)}{u}(k_1, k_2-\sigma)$, (5.36) becomes

$$v(k_1,k_2) = \sum_{i=0}^{m}\sum_{j=i}^{m} x_j F_{n+i-j} v(k_1, k_2-(n+i)\ell_2) + \gamma \underset{x(0,\ell_2)}{\overset{-1}{\Delta}} \underset{x(\pm\ell_1)}{u}(k_1, k_2-\sigma). \tag{5.38}$$

The proof of (5.37) follows from the relation,

$$\underset{x(0,\ell_2)}{\overset{-1}{\Delta}} \underset{x(\pm\ell_1)}{u}(k_1,k_2) = \sum_{i=1}^{n} F_i \underset{x(\pm\ell_1)}{u}(k_1-r(0), k_2-i\ell_2) \text{ and using (5.38).} \qquad \square$$

Theorem 5.4.2. *Considering (5.36), and denoting* $v(k_1\pm\ell_1,*) = v(k_1+\ell_1,*) + v(k_1-\ell_1,*)$ *and* $v(k_1\pm 2\ell_1,*) = v(k_1+2\ell_1,*) + v(k_1-2\ell_1,*)$, *we obtain the four types solutions for (5.36) as given below.*

(a).
$$\begin{aligned} v(k_1,k_2) &= {x_1}^n v(k_1, k_2-n\ell_2) - \sum_{i=1}^{n} \gamma {x_1}^i v(k_1\pm\ell_1, k_2-\sigma-(i-1)\ell_2) \\ &+ \sum_{i=1}^{n} \gamma {x_1}^{i-1} v(k_1, k_2-\sigma-(i-1)\ell_2) + \sum_{r=2}^{p} \left\{ \sum_{i=1}^{n} x_r {x_1}^{i-1} \Big[v(k_1, k_2-r\ell_2) \right. \\ &\left. -\gamma v(k_1\pm r\ell_1, k_2-\sigma-(i-1)\ell_2) \Big] \right\}, \end{aligned} \tag{5.39}$$

(b).
$$\begin{aligned} v(k_1,k_2) &= \frac{1}{{x_1}^n} v(k_1, k_2+n\ell_2) - \sum_{i=1}^{n} \frac{\gamma}{x_1^i} \{k_1, k_2-\sigma+i\ell_2\} \\ &+ \sum_{i=1}^{n} \frac{\gamma}{x_1^{i-1}} \{k_1\pm\ell_1, k_2-(i-1)\sigma+i\ell_2\} - \sum_{r=2}^{p} \left\{ \sum_{i=1}^{n} \frac{x_r}{{x_1}^i} \Big[v(k_1, k_2+(i-r)\ell_2) \right. \\ &\left. -\gamma v(k_1\pm r\ell_1, k_2-\sigma+i\ell_2) \Big] \right\}, \end{aligned} \tag{5.40}$$

(c). $$v(k_1,k_2) = \frac{1}{\gamma^n} v(k_1 - n\ell_1, k_2 + n\sigma - n\ell_2)$$
$$- \sum_{i=1}^{n} \frac{1}{\gamma^{i-1}} v(k_1 - (i+1)\ell_1, k_2 + (i-1)\sigma - (i-1)\ell_2)$$
$$- \sum_{i=1}^{n} \frac{1}{x_1\gamma^{i}} v(k_1 - i\ell_1, k_2 + i\sigma - (i-1)\ell_2)$$
$$+ \sum_{i=1}^{n} \frac{1}{x_1\gamma^{i-1}} v(k_1 - i\ell_1, k_2 + (i-1)\sigma - (i-1)\ell_2)$$
$$- \sum_{r=2}^{p} \left\{ \sum_{i=1}^{n} \frac{x_r}{x_1\gamma^{i}} \Big[v(k_1 - (i-r)\ell_1, k_2 + (i-1)\sigma - (i-1)\ell_2) \right.$$
$$\left. + v(k_1 - (r+i)\ell_1, k_2 + (i-1)\sigma - (i-1)\ell_2) \Big] \right\}$$
$$+ \sum_{r=2}^{p} \left\{ \sum_{i=1}^{n} \frac{x_r}{x_1\gamma^{i}} v(k_1 - i\ell_1, k_2 + i\sigma - (i+(r-1))\ell_2) \right\}, \tag{5.41}$$

(d). $$v(k_1,k_2) = \frac{1}{\gamma^n} v(k_1 + n\ell_1, k_2 - n\ell_2)$$
$$- \sum_{i=1}^{n} \frac{1}{\gamma^{i-1}} v(k_1 + (i+1)\ell_1, k_2 + (i-1)\sigma - (i-1)\ell_2)$$
$$- \sum_{i=1}^{n} \frac{1}{x_1\gamma^{i}} v(k_1 + i\ell_1, k_2 + i\sigma - (i-1)\ell_2)$$
$$+ \sum_{i=1}^{n} \frac{1}{x_1\gamma^{i-1}} v(k_1 + i\ell_1, k_2 + (i-1)\sigma - (i-1)\ell_2)$$
$$- \sum_{r=2}^{p} \left\{ \sum_{i=1}^{n} \frac{x_r}{x_1\gamma^{i-1}} \Big[v(k_1 + (i+r)\ell_1, k_2 + (i-1)\sigma - (i-1)\ell_2) \right.$$
$$\left. + v(k_1 + (r+i)\ell_1, k_2 + (i-1)\sigma - (i-1)\ell_2) \Big] \right\}$$
$$+ \sum_{r=2}^{p} \left\{ \sum_{i=1}^{n} \frac{x_r}{x_1\gamma^{i}} v(k_1 + i\ell_1, k_2 + i\sigma - (i+(r-1))\ell_2) \right\}. \tag{5.42}$$

Proof. (a). Applying the corresponding difference operators (5.36), we get

$$v(k_1,k_2) = x_1 v(k_1, k_2 - \ell_2) + \ldots + x_r v(k_1, k_2 - r\ell_2)$$
$$- x_1\gamma v(k_1 \pm \ell_1, k_2) - \ldots - x_r\gamma v(k_1 \pm r\ell_1, k_2). \tag{5.43}$$

Replacing k_2 by $k_2-\ell_2, k_2-2\ell_2, ..., k_2-n\ell_2$, we get the proof.

(b). $$v(k_1,k_2)=\frac{1}{x_1}v(k_1,k_2+\ell_2)+\gamma v(k_1\pm\ell_1,k_2-\sigma+\ell_2)-\frac{\gamma}{x_1}v(k_1,k_2-\sigma+\ell_2)$$
$$-\frac{x_2}{x_1}\Big[v(k_1,k_2-\ell_2)-\gamma v(k_1\pm 2\ell_1,k_2-\sigma+\ell_2)\Big]$$
$$-...-\frac{x_{(r-1)}}{x_1}\Big[v(k_1,k_2-\ell_2)-\gamma v(k_1\pm(r-1)\ell_1,k_2+\ell_2)\Big]$$
$$-\frac{x_r}{x_1}\Big[v(k_1,k_2-\ell_2)-\gamma v(k_1\pm r\ell_1,k_2-\sigma+\ell_2)\Big]. \tag{5.44}$$

When changing k_2 by $k_2+\ell_2, k_2+2\ell_2, ..., k_2+n\ell_2$ repeatedly, we get the result.

(c). The expression (5.36) becomes
$$v(k_1,k_2)=\frac{1}{\gamma}v(k_1-\ell_1,k_2+\sigma-\ell_2)-v(k_1-2\ell_1,k_2)+\frac{1}{x_1}v(k_1-\ell_1,k_2)$$
$$+\frac{x_2}{x_1\gamma}v(k_1-\ell_1,k_2+\sigma-2\ell_2)+\frac{x_3}{x_1\gamma}v(k_1-\ell_1,k_2+\sigma-3\ell_2)$$
$$+...+\frac{x_r}{x_1\gamma}v(k_1-\ell_1,k_2+\sigma-r\ell_2)$$
$$-\frac{x_2}{x_1}\Big[v(k_1+\ell_1,k_2)+v(k_1-3\ell_1,k_2)\Big]-\frac{x_3}{x_1}\Big[v(k_1+2\ell_1,k_2)+v(k_1-4\ell_1,k_2)\Big]$$
$$-...-\frac{x_r}{x_1}\Big[v(k_1+(r-1)\ell_1,k_2)+v(k_1-(r+1)\ell_1,k_2)\Big]-\frac{1}{x_1\gamma}v(k_1-\ell_1,k_2+\sigma).$$

Replacing k_1 by $k_1-\ell_1, k_1-2\ell_1, ..., k_1-n\ell_1$ and k_2 by $k_2-\ell_2, k_2-2\ell_2, ..., k_2-n\ell_2$ repeatedly, we get the proof.

(d). $$v(k_1,k_2)=\frac{1}{\gamma}v(k_1+\ell_1,k_2+\sigma-\ell_2)-v(k_1+2\ell_1,k_2)+\frac{1}{x_1}v(k_1+\ell_1,k_2)$$
$$+\frac{x_2}{x_1\gamma}v(k_1+\ell_1,k_2+\sigma-2\ell_2)+\frac{x_3}{x_1\gamma}v(k_1+\ell_1,k_2+\sigma-3\ell_2)+...+\frac{x_r}{x_1\gamma}v(k_1+\ell_1,k_2-r\ell_2)$$
$$-\frac{x_2}{x_1}\Big[v(k_1+3\ell_1,k_2)+v(k_1-\ell_1,k_2)\Big]-\frac{x_3}{x_1}\Big[v(k_1+4\ell_1,k_2)+v(k_1-2\ell_1,k_2)\Big]$$
$$-...-\frac{x_r}{x_1}\Big[v(k_1+(r+1)\ell_1,k_2)+v(k_1-(r_1)\ell_1,k_2)\Big]-\frac{1}{x_1\gamma}v(k_1+\ell_1,k_2+\sigma).$$

By changing k_1 by $k_1+\ell_1, k_1+2\ell_1, ..., k_1+n\ell_1$ and k_2 by $k_2-\ell_2, k_2-2\ell_2, ..., k_2-n\ell_2$

repeatedly, we get the proof. □

Example 5.4.3. *Suppose $v(k_1, k_2) = e^{k_1+k_2}$ is an exact solution of (5.36), then we have the relation* $\underset{x(0,\ell_2)}{\Delta} e^{k_1+k_2} = \gamma\big[\underset{x(\ell_1)}{\Delta} e^{k_1+k_2-\sigma} + \underset{x(-\ell_1)}{\Delta} e^{k_1+k_2-\sigma}\big]$*, which yields*

$$e^{k_1+k_2} - x_1 e^{k_1+k_2-\ell_2} - ... - x_r e^{k_1+k_2-r\ell_2}$$
$$= \gamma\big[e^{k_1+k_2-\sigma} - x_1 e^{k_1\pm\ell_1+k_2-\sigma} - ... - x_r e^{k_1\pm r\ell_1+k_2-\sigma}\big].$$

Cancelling $e^{k_1+k_2}$ on both sides we derive

$$\gamma = \frac{1 - x_1 e^{-\ell_2} - ... - x_r e^{-r\ell_2}}{e^{-\sigma} - x_1(e^{\ell_1-\sigma} + e^{-\ell_1-\sigma}) - ... - x_r(e^{r\ell_1-\sigma} + e^{-r\ell_1-\sigma})}. \tag{5.45}$$

When $m = 1$, $p = 2$, $k_1 = 1$, $\ell_1 = 1$, $k_2 = 2$, $\ell_2 = 2$, $x_1 = 1$, $x_2 = 2$, $v(k_1, k_2) = e^{(k_1+k_2)}$ and γ is as given in (5.44).

The MATLAB coding for (a) of Theorem 5.4.2 is as follows:

$exp(1+2) = exp(1)+symsum((0.1313589496.*(exp(3-((i-1).*2))+exp(1-((i-1).*2)))), i, 1, 1)+symsum((-0.1313589496.*exp(2-((i-1).*2))), i, 1, 1)+symsum((2.*(exp(-1) + 0.131359496.*(exp(4-((i-1).*2)) + exp(0-((i-1).*2))))), i, 1, 1).$

5.4.2 Temperature formula for thin plate

In this section, several types of solutions of equation (5.4) are arrived.

By Newton's law of cooling, the second order partial difference equation

$$\underset{x(0,0,\ell_3)}{\Delta} v(k_1, k_2, k_3) = \gamma\Big[\underset{x(\ell_1,\ell_2,0)}{\Delta} v(k) + \underset{x(-\ell_1,-\ell_2,0)}{\Delta} v(k)\Big], x = (x_1, x_2) \tag{5.46}$$

represents discrete heat equation of a thin plate. Here γ is Fibonacci heat conductivity of plate. The operators used in the R.H.S of (5.46) can be denoted as $\underset{x(\pm\ell_{1,2})}{\Delta}$. The Fibonacci heat equation (5.15) can be expressed as

$$\underset{x(0,0,\ell_3)}{\Delta} v(k) = \gamma \underset{x(\pm\ell_{1,2})}{\Delta} v(k); \quad x = (x_1, x_2), \tag{5.47}$$

where $\underset{x(\pm\ell_{1,2})}{\Delta} = \underset{x(\ell_1,0)}{\Delta} + \underset{x(-\ell_1,0)}{\Delta} + \underset{x(\ell_2,0)}{\Delta} + \underset{x(-\ell_2,0)}{\Delta}$, $(\ell_i, 0)$ means that i^{th} component is ℓ_i, remaining components are zero.

Theorem 5.4.4. *In (5.46), if $\underset{x(\pm\ell_{1,2})}{\Delta} v(k)$ are known functions, say $\underset{\pm}{u}(k)$, then*

$$v(k) = F_{n+1}v(k-(n+1)\ell_3) + x_2F_nv(k-(n+2)\ell_3) + \gamma\sum_{s=0}^{n} F_s \underset{\pm}{u}(k-s\ell_3). \tag{5.48}$$

Proof. The proof of (5.48) follows by taking $m = n = 3$ in (5.24) and equating the result with assuming that $\underset{x(\pm\ell_{1,2})}{\Delta} v(k) = \underset{\pm}{u}(k)$ in (5.46). $\square$

The following notations are used in Theorem 5.4.5,

$v(k-(\pm\ell_1, 0, s\ell_3)) = v(k-(\ell_1, 0, s\ell_3)) + v(k-(-\ell_1, 0, s\ell_3))$

$v(k-(\ell_1, \pm\ell_2, \ell_3)) = v(k-(\ell_1, \ell_2, \ell_3)) + v(k-(\ell_1, -\ell_2, \ell_3))$.

Theorem 5.4.5. *If $x_1\gamma \neq 0$ and $(1-4\gamma) \neq 0$ then equation (5.4) has solutions:*

$$\text{(a).}\ v(k) = \frac{{x_1}^m}{(1-4\gamma)^m}v(k-m\ell_3) - \sum_{s=1}^{m}\frac{\gamma {x_1}^s}{(1-4\gamma)^s}\Big[v(k-(\pm\ell_1, 0, (s-1)\ell_3)) + v(k-(0, \pm\ell_2, (s-1)\ell_3))\Big] + \sum_{s=1}^{m}\frac{x_2{x_1}^{s-1}}{(1-4\gamma)^s}\Big[v(k-(0,0,(s+1)\ell_3)) - \gamma\Big\{v(k-(\pm 2\ell_1, 0, (s-1)\ell_3)) + v(k-(0, \pm 2\ell_2, (s-1)\ell_3))\Big\}\Big], \tag{5.49}$$

$$\text{(b). } v(k) = \frac{(1-4\gamma)^m}{{x_1}^m} v(k+(0,0,m\ell_3)) + \sum_{s=1}^{m} \frac{\gamma(1-4\gamma)^{s-1}}{{x_1}^{s-1}} \Big[v(k+(\pm\ell_1,0,s\ell_3))$$
$$+ v(k+(0,\pm\ell_2,s\ell_3)) \Big] - \sum_{s=1}^{m} \frac{x_2(1-4\gamma)^{s-1}}{{x_1}^{s}} \Big[v(k+(0,0,(s-2)\ell_3))$$
$$- \gamma \Big\{ v(k+(\pm 2\ell_1,0,s\ell_3)) + v(k+(0,\pm 2\ell_2,s\ell_3)) \Big\} \Big], \tag{5.50}$$

$$\text{(c). } v(k) = \frac{1}{\gamma^m} v(k-(m\ell_1,0,m\ell_3)) + \sum_{r=1}^{m} \frac{x_2}{x_1\gamma^s} v(k-(s\ell_1,0,(s+1)\ell_3))$$
$$- \sum_{s=0}^{m} \frac{1}{\gamma^{s-1}} \Big[v(k-((s+1)\ell_1,0,(s-1)\ell_3)) + v(k-(s\ell_1 \pm \ell_2,(s-1)\ell_3)) \Big]$$
$$- \sum_{r=1}^{m} \frac{x_2}{x_1\gamma^{s-1}} \Big[v(k-((s-2)\ell_1,0,(s-1)\ell_3)) + v(k-((s+2)\ell_1,0,(s-1)\ell_3))$$
$$+ v(k-(s\ell_1,\pm 2\ell_2,(s-1)\ell_3)) \Big] - \sum_{s=0}^{m} \frac{(1-4\gamma)}{x_1\gamma^s} v(k-(s\ell_1,0,(s-1)\ell_3)), \tag{5.51}$$

$$\text{(d). } v(k) = \frac{1}{\gamma^m} v(k+(m\ell_1,0,-m\ell_3)) + \sum_{r=1}^{m} \frac{x_2}{x_1\gamma^s} v(k+(s\ell_1,0,-(s+1)\ell_3))$$
$$- \sum_{s=0}^{m} \frac{1}{\gamma^{s-1}} \Big[v(k+((s+1)\ell_1,0,-(s-1)\ell_3)) + v(k+(s\ell_1 \pm \ell_2,-(s-1)\ell_3)) \Big]$$
$$- \sum_{r=1}^{m} \frac{x_2}{x_1\gamma^{s-1}} \Big[v(k+((s+2)\ell_1,0,-(s-1)\ell_3)) + v(k+((s-2)\ell_1,0,-(s-1)\ell_3))$$
$$+ v(k+(s\ell_1,\pm 2\ell_2,-(s-1)\ell_3)) \Big] - \sum_{s=0}^{m} \frac{(1-4\gamma)}{x_1\gamma^s} v(k+(s\ell_1,0,-(s-1)\ell_3)). \tag{5.52}$$

Proof. (a). From (5.1) and (5.4), it is obvious to obtain

$$v(k) = \frac{x_1}{(1-4\gamma)} v(k+(0,0,-\ell_3)) + \frac{x_2}{(1-4\gamma)} v(k+(0,0,-2\ell_3))$$
$$- \frac{x_1\gamma}{(1-4\gamma)} v(k+(\pm\ell_1,0,0)) - \frac{x_2\gamma}{(1-4\gamma)} v(k+(\pm 2\ell_1,0,0))$$
$$- \frac{x_1\gamma}{(1-4\gamma)} v(k+(0,\pm\ell_2,0)) - \frac{x_2\gamma}{(1-4\gamma)} v(k+(0,\pm 2\ell_2,0)). \tag{5.53}$$

Replacing k_3 by $k_3 - \ell_3, k_3 - 2\ell_3, ..., k_3 - m\ell_3$ in (5.53) gives expressions for $v(k+(0,0,k_3 - r\ell_3))$ and $v(k \pm (\ell_1, 0, k_3 - r\ell_3)$, which yields (5.53).

(b). The heat equation (5.4) generates the relation

$$v(k) = \frac{(1-4\gamma)}{x_1} v(k(0, 0+\ell_3)) + \gamma v(k + (\pm\ell_1, 0, +\ell_3))$$
$$- \frac{x_2}{x_1} v(k + (0, 0, -\ell_3)) - \frac{x_2\gamma}{x_1}\Big[v(k + (\pm 2\ell_1, 0, \ell_3))\Big]. \tag{5.54}$$

The proof of (b) follows by replacing k_3 by $k_3 + \ell_3, k_3 + 2\ell_3, ..., k_3 + m\ell_3$ repeatedly in (5.54) and substituting corresponding γ-values in (5.54).

(c). Applying the Fibonacci difference operator (5.1) on (5.4) yields the relation

$$v(k) = \frac{1}{\gamma} v(k + (-\ell_1, 0, -\ell_3)) + \frac{x_2}{x_1\gamma} v(k + (-\ell_1, 0, -2\ell_3))$$
$$-\Big[v(k + (-2\ell_1, 0, 0)) + v(k + (-\ell_1 \pm \ell_2, 0)\Big] - \frac{(1-4\gamma)}{x_1\gamma} v(k + (-\ell_1, 0, 0))$$
$$-\frac{x_2}{x_1}\Big[v(k+(\ell_1, 0, 0)) + v(k+(-3\ell_1, 0, 0)) + v(k+(-\ell_1 \pm 2\ell_2, 0))\Big].$$

The proof of (c) follows by replacing k_1 by $k_1 - \ell_1, k_1 - 2\ell_1, ..., k_1 - m\ell_1$ and k_3 by $k_3 - \ell_3, k_3 - 2\ell_3, ..., k_3 - m\ell_3$ repeatedly and applying these in the above relation.

(d). By (5.1) the equation (5.4) gives the expression

$$v(k) = \frac{1}{\gamma} v(k + (\ell_1, 0, -\ell_3)) + \frac{x_2}{x_1\gamma} v(k + (\ell_1, 0, -2\ell_3))$$
$$-\Big[v(k+(2\ell_1, 0, 0)) + v(k+(\ell_1 \pm \ell_2, 0))\Big] - \frac{(1-4\gamma)}{x_1\gamma} v(k+(\ell_1, 0, 0))$$
$$-\frac{x_2}{x_1}\Big[v(k+(3\ell_1, 0, 0)) + v(k+(-\ell_1, 0, 0)) + v(k+(\ell_1 \pm 2\ell_2, 0))\Big],$$

which yields (d) by replacing k_1 by $k_1 + \ell_1, k_1 + 2\ell_1, ..., k_1 + m\ell_1$ and k_3 by $k_3 - \ell_3, k_3 - 2\ell_3, ..., k_3 - m\ell_3$ repeatedly. $\square$

The following example shows that the diffusion rate of thin plate can be identified if $v(k)$ in (5.4) is known and vice versa.

Example 5.4.6. *Suppose that* $v(k) = e^{k_1+k_2+k_3}$ *is a closed form solution of* (5.4)*, then it is obvious to obtain*

$\underset{x(0,\ell_3)}{\Delta} e^{k_1+k_2+k_3} = \gamma\big[\underset{x(\ell_{1,2})}{\Delta} e^{k_1+k_2+k_3} + \underset{x(-\ell_{1,2})}{\Delta} e^{k_1+k_2+k_3}\big]$*, which yields*

$$\gamma = \frac{1 - x_1e^{-\ell_3} - x_2e^{-2\ell_3}}{4 - x_1(e^{\ell_1} + e^{-\ell_1} + e^{\ell_2} + e^{-\ell_2}) - x_2(e^{2\ell_1} + e^{-2\ell_1} + e^{2\ell_2} + e^{-2\ell_2})}. \tag{5.55}$$

For numerical verification, we give the MATLAB coding for (a) of Theorem 5.2.2*. When* $m = 1$, $k_1 = 1$, $\ell_1 = 1$, $k_2 = 2$, $\ell_2 = 2$, $x_1 = 1$, $x_2 = 2$, $k_3 = 3$, $\ell_3 = 3$, $v(k_1, k_2, k_3) = e^{(k_1+k_2+k_3)}$ *and* γ *is as given in* (5.55)*, the code is as follows:*

$((1.\wedge 1)./(1.028886517.\wedge 1)).*exp(6-(1.*3))-(symsum((((-0.007221629292).*(1.\wedge i))./(1.028886517.\wedge (i))).*(exp(7-((i-1).*3))+exp(5-((i-1).*3))+(exp(8-((i-1).*3))+exp(4-((i-1).*3)))),i,1,1))+(symsum(((2.*(1.\wedge (i-1)))./(1.028886517.\wedge i)).*(exp(6-((i+1).*3))+(0.007221629292.*((exp(8-((i-1).*3))+exp(4-((i-1).*3))+exp(10-((i-1).*3))+exp(2-((i-1).*3)))))),i,1,1)).$

For getting accuracy value of heat transmission, (5.1) can be replaced by,

$$\begin{aligned}\underset{x(0,\ell_3)}{\Delta} v(k) &= v(k) - x_1v(k + (0, 0, -\ell_3)) \\ &\quad - x_2v(k + (0, 0, -2\ell_3)) - x_3v(k + (0, 0, -3\ell_3)).\end{aligned} \tag{5.56}$$

In this case, the corresponding heat equation model becomes

$$\underset{x(0,\ell_3)}{\Delta} v(k) = \gamma \underset{x(\pm\ell_{1,2})}{\Delta} v(k); \quad x = (x_1, x_2, x_3). \tag{5.57}$$

Theorem 5.4.7. *Assume that there exists a positive integer* n*, and a real number* $\ell_2 > 0$ *such that* $v(k + (0, 0, -n\ell_3))$ *and* $\underset{x(\pm\ell_{1,2})}{\Delta} v(k) = \underset{x(\pm\ell_{1,2})}{u}(k)$ *are known. Then,* *the heat equation* (5.57) *has a solution* $v(k)$ *of the form*

$$v(k) = F_{n+1}v(k+(0,0,-(n+1)\ell_3)) + (x_2F_n + x_3F_{n-1})v(k+(0,0,-(n+2)\ell_3))$$

$$+ x_3F_nv(k+(0,0,-(n+3)\ell_3)) + \gamma\sum_{s=0}^{n} F_s \underset{x(\pm\ell_{1,2})}{u}(k+(0,0,-s\ell_3)), \tag{5.58}$$

where $F_0 = 1,\ F_1 = x_1$ *and* $F_n = x_1F_{n+2} + x_2F_{n+1} + x_3F_n$.

Theorem 5.4.8. *Consider the equation* (5.57). *Then, the following four types solutions of the equation* (5.57) *are equivalent:*

$$\begin{aligned}
\text{(a). } v(k) = {} & \frac{{x_1}^m}{(1-4\gamma)^m}v(k+(0,0,-m\ell_3)) - \sum_{s=1}^{m}\frac{\gamma {x_1}^s}{(1-4\gamma)^s}\Big[v(k+(\pm\ell_1,0,-(s-1)\ell_3)) \\
& + v(k+(0,\pm\ell_2,-(s-1)\ell_3))\Big] + \sum_{s=1}^{m}\frac{x_2{x_1}^{s-1}}{(1-4\gamma)^s}\Big[v(k+(0,0,-(s+1)\ell_3)) \\
& -\gamma\left[v(k+(\pm2\ell_1,0,-(s-1)\ell_3)) + v(k+(0,\pm2\ell_2,-(s-1)\ell_3))\right]\Big] \\
& + \sum_{s=1}^{m}\frac{x_3{x_1}^{s-1}}{(1-4\gamma)^s}\Big[v(k+(0,0,-(s+2)\ell_3)) - \gamma\big[v(k+(\pm3\ell_1,0,-(s-1)\ell_3)) \\
& + v(k+(0,\pm3\ell_2,-(s-1)\ell_3))\big]\Big],
\end{aligned} \tag{5.59}$$

$$\begin{aligned}
\text{(b). } v(k) = {} & \frac{(1-4\gamma)^m}{{x_1}^m}v(k(0,0,+m\ell_3)) + \sum_{s=1}^{m}\frac{\gamma(1-4\gamma)^{s-1}}{{x_1}^{s-1}}\Big[v(k+(\pm\ell_1,0,+s\ell_3)) \\
& + v(k+(0,\pm\ell_2,+s\ell_3))\Big] - \sum_{s=1}^{m}\frac{x_2(1-4\gamma)^{s-1}}{{x_1}^s}\Big[v(k+(0,0,(s-2)\ell_3)) \\
& -\gamma[v(k+(\pm2\ell_1,0,s\ell_3)) + v(k+(0,\pm2\ell_2,s\ell_3))]\Big] \\
& - \sum_{s=1}^{m}\frac{x_3(1-4\gamma)^{s-1}}{{x_1}^s}\Big[v(k+(0,0,(s-3)\ell_3)) \\
& - \gamma[v(k+(\pm3\ell_1,0,s\ell_3)) + v(k+(0,\pm3\ell_2,s\ell_3))]\Big],
\end{aligned} \tag{5.60}$$

$$\text{(c). } v(k) = \frac{1}{\gamma^m}v(k+(-m\ell_1,0,-m\ell_3)) + \sum_{r=1}^{m}\frac{x_2}{x_1\gamma^s}v(k+(-s\ell_1,0,-(s+1)\ell_3))$$

$$+\sum_{r=1}^{m}\frac{x_3}{x_1\gamma^s}v(k+(-s\ell_1,0,-(s+2)\ell_3))-\sum_{s=0}^{m}\frac{(1-4\gamma)}{x_1\gamma^s}v(k+(-s\ell_1,0,-(s-1)\ell_3))$$
$$-\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^{s-1}}\Big[v(k+(-(s-2)\ell_1,0,-(s-1)\ell_3)+v(k+(-(s+2)\ell_1,0,-(s-1)\ell_3))$$
$$+v(k+(-s\ell_1\pm 2\ell_2,-(s-1)\ell_3))\Big]-\sum_{r=1}^{m}\frac{x_3}{x_1\gamma^{s-1}}\Big[v(k+(-(s-3)\ell_1,0,-(s-1)\ell_3))$$
$$+v(k+(-(s+3)\ell_1,0,-(s-1)\ell_3))+v(k+(-s\ell_1\pm 3\ell_2,-(s-1)\ell_3))\Big]$$

$$-\sum_{s=0}^{m}\frac{1}{\gamma^{s-1}}\Big[v(k+(-(s+1)\ell_1,0,-(s-1)\ell_3))+v(k+(-s\ell_1\pm\ell_2,0,-(s-1)\ell_3))\Big], \quad (5.61)$$

(d). $$v(k)=\frac{1}{\gamma^m}v(k+(m\ell_1,0,-m\ell_3))+\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^s}v(k+(s\ell_1,0,-(s+1)\ell_3))$$
$$+\sum_{r=1}^{m}\frac{x_3}{x_1\gamma^s}v(k+(s\ell_1,0,-(s+2)\ell_3))-\sum_{s=0}^{m}\frac{(1-4\gamma)}{x_1\gamma^s}v(k+(s\ell_1,0,-(s-1)\ell_3))$$
$$-\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^{s-1}}\Big[v(k+((s+2)\ell_1,0,-(s-1)\ell_3))+v(k+((s-2)\ell_1,0,(s-1)\ell_3))$$
$$+v(k+(s\ell_1\pm 2\ell_2,-(s-1)\ell_3))\Big]-\sum_{r=1}^{m}\frac{x_3}{x_1\gamma^{s-1}}\Big[v(k+((s+3)\ell_1,-(s-1)\ell_3))$$
$$+v(k+((s-3)\ell_1,-(s-1)\ell_3))+v(k+(s\ell_1\pm 3\ell_2,-(s-1)\ell_3))\Big]$$

$$-\sum_{s=0}^{m}\frac{1}{\gamma^{s-1}}\Big[v(k+((s+1)\ell_1,0,-(s-1)\ell_3))+v(k+(s\ell_1\pm\ell_2,-(s-1)\ell_3))\Big]. \quad (5.62)$$

Proof. The proof of this theorem is similar to that of Theorem 5.2.2. □

The following theorem focuses on heat equation formula with higher order Fibonacci numbers.

Theorem 5.4.9. *Four types of solutions of the equation* (5.35) *are given by*

(a). $$v(k)=\frac{{x_1}^n}{(1-4\gamma)^n}v(k-(0,0,n\ell_3)-\sum_{s=1}^{n}\frac{\gamma {x_1}^s}{(1-4\gamma)^s}\{v(k-(\pm\ell_1,0,(s-1)\ell_3))$$

$$+v(k-(0,\pm\ell_2,(s-1)\ell_3))\}+\sum_{r=2}^{n}\left\{\sum_{s=1}^{r}\frac{x_r{x_1}^{s-1}}{(1-4\gamma)^s}\Big[v(k-(0,0,(s+(r-1)))\ell_3)\right.$$

$$\left.-\gamma v(k-(\pm r\ell_1,(s-1)\ell_2,0))+v(k-(0,\pm r\ell_2,(s-1)\ell_3))\Big]\right\}, \quad (5.63)$$

(b).
$$\begin{aligned} v(k) = {} & \frac{(1-4\gamma)^n}{{x_1}^n}v(k+(0,0,n\ell_3)) \\ & -\sum_{s=1}^{n}\frac{\gamma(1-4\gamma)^{s-1}}{{x_1}^{s-1}}\Big[v(k+(\pm\ell_1,0,s\ell_3)+v(k+(0,\pm\ell_2,s\ell_3))\Big] \\ & -\sum_{r=2}^{n}\left\{\sum_{s=1}^{r}\frac{x_r(1-4\gamma)^{s-1}}{{x_1}^s}\Big[v(k+(0,0,(s-r)\ell_3))\right. \\ & \left.-\gamma v(k+(\pm r\ell_1,0,s\ell_3))+v(k+(0,\pm r\ell_2,s\ell_3))\Big]\right\}, \end{aligned} \quad (5.64)$$

(c).
$$\begin{aligned} v(k) = {} & \frac{1}{\gamma^n}v(k-(n\ell_1,0,n\ell_3))-\sum_{s=0}^{n}\frac{1}{\gamma^{s-1}}\Big[v(k-((s+1)\ell_1,0,(s-1)\ell_3)) \\ & +v(k-(s\ell_1,0,(s-1)\ell_3))\Big]-\sum_{s=0}^{n}\frac{(1-4\gamma)}{x_1\gamma^s}v(k-(s\ell_1,0,(s-1)\ell_3)) \\ & -\sum_{r=2}^{n}\left\{\sum_{s=1}^{r}\frac{x_r}{x_1\gamma^{s-1}}\Big[v(k-(s\ell_1,0,(s+(r-1)\ell_2))\Big]\right\} \\ & +\sum_{r=2}^{n}\left\{\sum_{s=1}^{r}\frac{x_r}{x_1\gamma^s}v(k-((s-r)\ell_1,0,(s-1)\ell_3))\right. \\ & \left.+v(k-((s+r)\ell_1,0,(s-1)\ell_3))+v(k-(s\ell_1,\pm r\ell_2,(s-1)\ell_3))\right\}, \end{aligned} \quad (5.65)$$

(d).
$$\begin{aligned} v(k_) = {} & \frac{1}{\gamma^n}v(k+(n\ell_1,0,-n\ell_3))-\sum_{s=0}^{n}\frac{1}{\gamma^{s-1}}\Big[v(k+((s+1)\ell_1,0,-(s-1)\ell_3)) \\ & +v(k+(s\ell_1,0,-(s-1)\ell_3))\Big]-\sum_{s=0}^{n}\frac{(1-4\gamma)}{x_1\gamma^s}v(k+(s\ell_1,0,-(s-1)\ell_3)) \\ & -\sum_{r=2}^{n}\left\{\sum_{s=1}^{r}\frac{x_r}{x_1\gamma^{s-1}}\Big[v(k+(s\ell_1,0,-(s+(r-1)\ell_2))\Big]\right\} \\ & +\sum_{r=2}^{n}\left\{\sum_{s=1}^{r}\frac{x_r}{x_1\gamma^s}v(k+((s-r)\ell_1,0,-(s-1)\ell_3))\right. \\ & \left.+v(k+((s+r)\ell_1,0,-(s-1)\ell_3))+v(k+(s\ell_1,\pm r\ell_2,-(s-1)\ell_3))\right\}. \end{aligned} \quad (5.66)$$

Proof. The Proof is as similar as Theorem 5.2.2. □

5.4.3 Fibonacci heat equation of medium

Heat equation model for medium is obtained by extending the theory of thin plate with three variables into five variables $k = (k_1, k_2, k_3, k_4, k_5)$ where (k_1, k_2, k_3) denote the point, and k_4, k_5 denote the time and density of the medium. By (5.1) and Newton law of cooling, the partial difference equation of medium is

$$\underset{x(\ell_{4,5})}{\Delta} v(k) = \gamma \underset{x(\pm\ell_{1,2,3})}{\Delta} v(k); \quad x = (x_1, x_2), \tag{5.67}$$

where $\underset{x(\pm\ell_{1,2,3})}{\Delta} = \underset{x(\ell_1,0)}{\Delta} + \underset{x(-\ell_1,0)}{\Delta} + \underset{x(\ell_2,0)}{\Delta} + \underset{x(-\ell_2,0)}{\Delta} + \underset{x(\ell_3,0)}{\Delta} + \underset{x(-\ell_3,0)}{\Delta}$, $(\ell_i, 0)$ means that i^{th} component is ℓ_i, remaining components are zero.

In general, (5.56) can be extended to m-tuple $x = (x_1, x_2, ..., x_m)$ as

$$\underset{x(\ell)}{\Delta} v(k) = x_1 v(k - \ell) - x_2 v(x - 2\ell) - ... - x_m v(k - m\ell)$$

and (5.67) becomes

$$\underset{x(\ell_{4,5})}{\Delta} v(k) = \gamma \underset{x(\pm\ell_{1,2,3})}{\Delta} v(k); \quad x = (x_1, x_2, ..., x_m). \tag{5.68}$$

Theorem 5.4.10. *If* $\underset{x(\pm\ell_{1,2,3})}{\Delta} v(k) = \underset{x(\pm\ell_{1,2,3})}{u}(k)$ *are known, then*

$$v(k) = F_{n+1} v(k - (0, 0, 0, (n + 1)\ell_4, (n + 1)\ell_5))$$

$$+ x_2 F_n v(k - (0, 0, 0, (n + 2)\ell_4, (n + 2)\ell_5)) + \gamma \sum_{s=0}^{n} F_s \underset{x(\pm\ell_{1,2,3})}{u}(k - (0, 0, 0, s\ell_4, s\ell_5)) \tag{5.69}$$

Proof. Argument similar to Theorem 5.4.9 gives the proof of (5.69). □

Consider the following notations which will be used in the subsequent theorems:

$$v(k+(\pm\ell_1,*)) = v(k+(\ell_1,*)) + v(k+(-\ell_1,*)),$$

$$v(k+(\pm 2\ell_1,*)) = v(k+(2\ell_1,*)) + v(k+(-2\ell_1,*)),$$

$$v(k+(2\pm\ell_2,*)) = v(k+(\ell_2,*)) + v(k+(-\ell_2,*)),$$

$$v(k+(\pm 2\ell_2,*)) = v(k+(2\ell_2,*)) + v(k+(-2\ell_2,*)).$$

$$v(k+(\pm\ell_3,*)) = v(k+(+\ell_3,*)) + v(k+(-\ell_3,*)),$$

$$v(k+(\pm 2\ell_3,*)) = v(k+(2\ell_3,*)) + v(k+(-2\ell_3,*)).$$

$$v(k\pm\ell_{(1,2,3)},*) = v(k+\ell_{(1,2,3)},*) + v(k-\ell_{(1,2,3)},*),$$

$$v(k\pm 2\ell_{(1,2,3)},*) = v(k+2\ell_{(1,2,3)},*) + v(k-2\ell_{(1,2,3)},*).$$

$$v(k_2\pm\ell_2,*) = v(k_2+\ell_2,*) + v(k_2-\ell_2,*),$$

$$v(k_2\pm 2\ell_2,*) = v(k_2+2\ell_2,*) + v(k_2-2\ell_2,*).$$

Theorem 5.4.11. *The equation (5.67) has solutions of the form*

(a). $$\begin{aligned} v(k) = {} & \frac{{x_1}^m}{(1-6\gamma)^m} v(k-(0,0,0,m\ell_4,m\ell_5)) \\ & + \sum_{s=1}^{m} \frac{x_2 x_1^{s-1}}{(1-6\gamma)^s} \Big[v(k-(\pm\ell_1,0,0,(s-1)\ell_4,(s-1)\ell_5)) \Big] \\ & - \sum_{s=1}^{m} \frac{x_1^s\gamma}{(1-6\gamma)^s} \{ v(k-(\pm\ell_1,0,0,(s-1)\ell_4,(s-1)\ell_5)) \\ & + v(k-(0,\pm\ell_2,0,(s-1)\ell_4,(s-1)\ell_5)) \\ & + v(k-(0,0,\pm\ell_3,(s-1)\ell_4,(s-1)\ell_5)) \} \\ & - \sum_{s=1}^{m} \frac{x_2 x_1^{(s-1)}\gamma}{(1-6\gamma)^s} \{ v(k-(\pm 2\ell_1,0,0,(s-1)\ell_4,(s-1)\ell_5)) \end{aligned}$$

$$+v(k-(0,\pm 2\ell_2,0,(s-1)\ell_4,(s-1)\ell_5))$$

$$+v(k-(0,0,\pm 2\ell_3,-(s-1)\ell_4,-(s-1)\ell_5)\}\,, \tag{5.70}$$

(b). $$v(k)=\frac{(1-6\gamma)^m}{{x_1}^m}v(k+(0,0,0,m\ell_4,m\ell_5))$$
$$-\sum_{s=1}^{m}\frac{x_2(1-6\gamma)^{s-1}}{{x_1}^s}v(k+(0,0,0,(s-2)\ell_4,(s-2)\ell_5))$$
$$-\sum_{s=1}^{m}\frac{\gamma(1-6\gamma)^{s-1}}{{x_1}^{(s-1)}}\Big[v(k+(\pm\ell_1,\ell_2,\ell_3,s\ell_4,s\ell_5))$$
$$+\sum_{s=1}^{m}\frac{x_2\gamma(1-6\gamma)^{(s-1)}}{x_1^s}v(k+(\pm 2\ell_1,\ell_2,\ell_3,s\ell_4,s\ell_5)\Big], \tag{5.71}$$

(c). $$v(k)=\frac{1}{\gamma^m}v(k+(m\ell_1,0,0,-m\ell_4,-m\ell_5))$$
$$+\sum_{r=1}^{m}\frac{x_2}{x_1\gamma^s}v(k+(-s\ell_1,0,0,-(s+1)\ell_4,-(s+1)\ell_5))$$
$$-\sum_{s=1}^{m}\frac{1}{\gamma^{s-1}}\Big[v(k+(-(s+1)\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5))$$
$$+v(k+(-s\ell_1,\pm\ell_2,0,-(s-1)\ell_4,-(s-1)\ell_5))$$
$$+v(k+(-s\ell_1,0,\pm\ell_3,-(s-1)\ell_4,-(s-1)\ell_5))\Big]$$
$$-\sum_{s=1}^{m}\frac{x_2}{x_1\gamma^{s-1}}\Big[v(k+(-(s-2)\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5)$$
$$+v(k+(-(s+2)\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5))$$
$$+v(k+(-s\ell_1,\pm 2\ell_2,0,-(s-1)\ell_4,-(s-1)\ell_5))$$
$$+v(k-s\ell_1,0,\pm 2\ell_3,-(s-1)\ell_4,-(s-1)\ell_5))\Big]$$
$$-\sum_{s=1`}^{m}\frac{(1-6\gamma)}{x_1\gamma^s}v(k+(-s\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5)), \tag{5.72}$$

(d). $$v(k)=\frac{1}{\gamma^m}v(k+(m\ell_1,0,0,-m\ell_4,-m\ell_5))$$
$$+\sum_{s=1}^{m}\frac{x_2}{x_1\gamma^s}v(k+(s\ell_1,0,0,-(s+1)\ell_4,-(s+1)\ell_5))$$

$$
\begin{aligned}
&-\sum_{s=1}^{m}\frac{1}{\gamma^{s-1}}\Big[v(k+((s+1)\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5))\\
&+v(k+(s\ell_1,\pm\ell_2,0,-(s-1)\ell_4,-(s-1)\ell_5))\\
&+v(k+(s\ell_1,\pm\ell_2,0,-(s-1)\ell_4,-(s-1)\ell_5))\\
&+v(k+(s\ell_1,0,\pm\ell_3,-(s-1)\ell_4,-(s-1)\ell_5))\Big]\\
&-\sum_{s=1}^{m}\frac{x_2}{x_1\gamma^{s-1}}\Big[v(k+((s-2)\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5))\\
&+v(k+((s+2)\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5))\\
&+v(k+(s\ell_1,\pm 2\ell_2,0,-(s-1)\ell_4,-(s-1)\ell_5))\\
&+v(k+(s\ell_1,0,\pm 2\ell_3,-(s-1)\ell_4,-(s-1)\ell_5))\Big]
\end{aligned}
$$

$$
-\sum_{s=1}^{m}\frac{(1-6\gamma)}{x_1\gamma^{s}}\Big[v(k+(s\ell_1,-(s-1)\ell_4,+(s-1)\ell_5)). \tag{5.73}
$$

Proof. (a). Expansion of (5.67) directly generates the relation

$$
\begin{aligned}
v(k) = {}& \frac{x_1}{(1-6\gamma)}v(k+(0,0,0,-\ell_4,-\ell_5))+\frac{x_2}{(1-6\gamma)}v(k+(0,0,0,-2\ell_4,-2\ell_5))\\
&-\frac{x_1\gamma}{(1-6\gamma)}v(k+(\pm\ell_1,0,0,0,0))+v(k+(0,\pm\ell_2,0,0,0))\\
&+v(k+(0,0,\pm\ell_3,0,0))-\frac{x_2\gamma}{(1-6\gamma)}v(k+(\pm 2\ell_1,0,0,0,0))
\end{aligned}
$$

$$
+v(k+(0,\pm 2\ell_2,0,0,0))+v(k+(0,0,\pm 2\ell_3,0,0)). \tag{5.74}
$$

By replacing k_3 by $k_3-\ell_3, k_3-2\ell_3, ..., k_3-m\ell_3$ in (5.74), we obtain the proof.

(b). The heat equation (5.67) yields the relation

$$
\begin{aligned}
v(k) = {}& \frac{(1-6\gamma)}{x_1}v(k+(0,0,0,\ell_4,\ell_5)-\frac{x_2}{x_1}v(k+(0,0,0,-\ell_4,-\ell_5))\\
&+\gamma\Big[v(k+(\pm\ell_1,+\ell_4,+\ell_5))+v(k+(0,\pm\ell_2,0,\ell_4,\ell_5))\\
&\qquad +v(k+(0,0,\pm\ell_3,\ell_4,\ell_5))\Big]
\end{aligned}
$$

$$+\frac{x_2\gamma}{x_1}\left[v(k+(\pm 2\ell_1,0,0,\ell_4,\ell_5))+v(k+(0,\pm 2\ell_2,0,\ell_4,\ell_5)\right.$$

$$\left.+v(k+(0,0,\pm 2\ell_3,\ell_4,\ell_5))\right]. \tag{5.75}$$

The proof of (b) follows by replacing k_3 by $k_3+\ell_3, k_3+2\ell_3, ..., k_3+m\ell_3$ repeatedly and substituting corresponding γ-values in (5.75).

(c). A simple calculation on (5.67) gives the expression

$$\begin{aligned}
v(k) = {} & \frac{1}{\gamma}v(k+(-\ell_1,0,0,-\ell_4,-\ell_5))+\frac{x_2}{x_1\gamma}v(k+(-\ell_1,0,0,-2\ell_4,-2\ell_5)) \\
& -\Big[v(k+(-2\ell_1,0,0,0,0)+v(k+(-\ell_1\pm\ell_2,0,0,0) \\
& +v(k+(-\ell_1\pm\ell_3,0,0,0))\Big]-\frac{x_2}{x_1}\Big[v(k+(-3\ell_1,0,0,0,0)) \\
& +v(k+(-\ell_1,0,0,0,0))+v(k+(-\ell_1\pm 2\ell_2,0,0,0)) \\
& +v(k+(-\ell_1,\pm 2\ell_3,0,0,0))\Big]-\frac{(1-6\gamma)}{x_1\gamma}v(k-\ell_1,0,0).
\end{aligned} \tag{5.76}$$

The proof of (c) follows by replacing k_1 by $k_1-\ell_1, k_1-2\ell_1, ..., k_1-m\ell_1$ and k_4 by $k_4-\ell_4, k_4-2\ell_4, ..., k_4-m\ell_4$ and k_5 by $k_5-\ell_5, k_5-2\ell_5, ..., k_5-m\ell_5$ repeatedly.

(d). (5.67) gives the expression

$$\begin{aligned}
v(k) = {} & \frac{1}{\gamma}v(k+(\ell_1,0,0,-\ell_4,-\ell_5))+\frac{x_2}{x_1\gamma}v(k+(\ell_1,0,0,-2\ell_4,-2\ell_5)) \\
& -\Big[v(k(+2\ell_1,0,0,0,0))+v(k+(\ell_1\pm\ell_2,0,0,0)) \\
& +v(k+(\ell_1\pm\ell_3,0,0,0))\Big]-\frac{x_2}{x_1}\Big[v(k+(3\ell_1,0,0,0,0) \\
& +v(k+(\ell_1,0,0,0,0))+v(k+(\ell_1\pm 2\ell_2,0,0,0)) \\
& +v(k+(\ell_1\pm 2\ell_3,0,0,0))\Big]-\frac{(1-6\gamma)}{x_1\gamma}v(k+(\ell_1,0,0,0,0)).
\end{aligned} \tag{5.77}$$

The proof of (d) follows by replacing k_1 by $k_1+\ell_1, k_1+2\ell_1, ..., k_1+m\ell_1$ and k_3 by

$k_4+\ell_4, k_4+2\ell_4, ..., k_4+m\ell_4$ and k_4 by $k_5+\ell_5, k_5+2\ell_5, ..., k_5+m\ell_5$ repeatedly. The proof and verification are similar as the Theorem 5.2.2. $\square$

Example 5.4.12. *The following example shows that the diffusion rate of rod can be identified if the solution $v(k,k_3)$ of (5.67) is known and vice versa. Suppose that $v(k,k_4,k_5)=e^{k+k_4+k_5}$ is a closed form solution of (5.67), then we have the relation $\underset{0,\ell_2(x)}{\Delta} e^{k+k_4+k_5}=\gamma\big[\underset{\ell_1(x)}{\Delta} e^{k+k_4+k_5}+\underset{-\ell_1(x)}{\Delta} e^{k+k_4+k_5}\big]$, which yields*

Cancelling $e^{k+k_4+k_5}$ on both sides derives

$$\gamma=\frac{1-x_1(e^{-\ell_4}+e^{-\ell_5})-x_2(e^{-2\ell_4}+e^{2\ell_5})}{6-x_1(a+b+c)-x_2(2a+2b+2c)}, \tag{5.78}$$

where $a=e^{\ell_1}+e^{-\ell_1}, b=e^{\ell_2}+e^{-\ell_2}, c=e^{\ell_3}+e^{-\ell_3}$.

For numerical verification, we give the MATLAB coding for (a) of Theorem 5.2.2. When $m=5$, $k_1=1$, $\ell_1=1$, $k_2=1$, $\ell_2=1$, $x_1=1$, $x_2=1$, $k_3=\ell_3=k_4=\ell_4=k_5=\ell_5=2$, $v(k_1,k_2,k_3)=e^{(k_1+k_2+k_3)}$ and γ is as given in (5.72), the code is as follows:

$(((1.\wedge 5)./(1.076110962).\wedge 5).*exp(8-(2.*5)-(2.*5)))+(symsum(((((1.*((1).\wedge (s-1)))./(1.076110962).\wedge s).*(exp(8-(2.*(s+1))-(2.*(s+1)))))-((((1.\wedge s).*(-0.01268516))./(1.076110962).\wedge s).*(exp(9-(2.*(s-1))-(2.*(s-1)))+exp(7-(2.*(s-1))-(2.*(s-1)))+exp(9-(2.*(s-1))-(2.*(s-1)))+exp(7-(2.*(s-1))-(2.*(s-1)))+exp(10-(2.*(s-1))-(2.*(s-1)))+exp(6-(2.*(s-1))-(2.*(s-1)))))-(((((1.\wedge (s-1)).*(-0.01268516))./(1.076110962).\wedge s).*(exp(10-(2.*(s-1))-(2.*(s-1)))+exp(6-(2.*(s-1))-(2.*(s-1)))+$

$exp(10-(2.*(s-1))-(2.*(s-1)))+exp(6-(2.*(s-1))-(2.*(s-1)))+exp(12-$

$(2.*(s-1))-(2.*(s-1)))+exp(4-(2.*(s-1))-(2.*(s-1)))))))), s, 1, 5)).$

For getting accuracy value of heat transmission, (5.1) can be replaced by,

$$\underset{x(0,\ell_3)}{\Delta} v(k,k_3) = v(k,k_3) - x_1 v(k,k_3-\ell_3) \\ - x_2 v(k,k_3-2\ell_3)) - x_3 v(k,k_3-3\ell_3). \tag{5.79}$$

In this case, the corresponding heat equation model is given by

$$\underset{x(0,\ell_3)}{\Delta} v(k_1,k_2) = \gamma \underset{x(\pm(\ell_1,\ell_2))}{\Delta} v(k_1,k_2); \quad x = (x_1,x_2,x_3). \tag{5.80}$$

As in the proof of Theorem 5.4.10, we get the following theorem.

Theorem 5.4.13. *Assume that there exists a positive integer n, and a real number $\ell_2 > 0$ such that $v(k,k_3-n\ell_3)$ and $\underset{x(\pm(\ell_1,\ell_2))}{\Delta} v(k,k_3) = \underset{x(\pm(\ell_1,\ell_2))}{u}(k,k_3)$ are known. Then, the heat equation (5.80) has a solution $v(k,k_3)$ of the form*

$$v(k,k_3) = F_{n+1} v(k,k_3-(n+1)\ell_3) + (x_2F_n + x_3F_{n-1}) v(k,k_3-(n+2)\ell_3) \\ + x_3F_n v(k,k_3-(n+3)\ell_3) + \gamma \sum_{i=0}^{n} F_i \underset{x(\pm(\ell_1,\ell_2))}{u}(k,k_3-i\ell_3) \tag{5.81}$$

where $F_0 = 1$, $F_1 = x_1$ and $F_n = x_1F_{n+2} + x_2F_{n+1} + x_3F_n$.

Let us use the following notations in the below theorem:

$v(k_1 \pm \ell_1, *) = v(k_1+\ell_1, *) + v(k_1-\ell_1, *)$, $v(k_1 \pm 2\ell_1, *) = v(k_1+2\ell_1, *) + v(k_1-2\ell_1, *)$,

$v(k_1 \pm 3\ell_1, *) = v(k_1+3\ell_1, *) + v(k_1-3\ell_1, *)$.

$v(k_1 \pm \ell_2, *) = v(k_1+\ell_2, *) + v(k_1-\ell_2, *)$, $v(k_1 \pm 2\ell_2, *) = v(k_1+2\ell_2, *) + v(k_1-2\ell_2, *)$,

$v(k_1 \pm 3\ell_2, *) = v(k_1+3\ell_2, *) + v(k_1-3\ell_2, *)$.

The following theorem gives more accuracy values of heat transmission.

Theorem 5.4.14. *Consider the equation (5.80). Then, the following four types solutions of the equation (5.80) are equivalent:*

(a).
$$
\begin{aligned}
v(k) = {} & \frac{{x_1}^m}{(1-6\gamma)^m} v(k+(0,0,0-m\ell_4, m\ell_5)) \\
& + \sum_{i=1}^m \frac{x_2 x_1^{i-1}}{(1-6\gamma)^i}\Big[v(k+(\pm\ell_1,0,0,-(i-1)\ell_4,-(i-1)\ell_5))\Big] \\
& - \sum_{i=1}^m \frac{x_1^i\gamma}{(1-6\gamma)^i}\Big\{v(k+(\pm\ell_1,0,0,-(i-1)\ell_4,-(i-1)\ell_5)) \\
& \qquad + v(k+(0,\pm\ell_2,0,-(i-1)\ell_4,-(i-1)\ell_5)) \\
& \qquad + v(k+(0,0,\pm\ell_3,-(i-1)\ell_4,-(i-1)\ell_5))\Big\} \\
& - \sum_{i=1}^m \frac{x_2 x_1^{(i-1)}\gamma}{(1-6\gamma)^i}\Big\{v+(k+(\pm 2\ell_1,0,0,-(i-1)\ell_4,-(i-1)\ell_5)) \\
& \qquad + v(k+(0,\pm 2\ell_2,0,-(i-1)\ell_4,-(i-1)\ell_5)) \\
& \qquad + v(k+(0,0,\pm 2\ell_3,-(i-1)\ell_4,-(i-1)\ell_5))\Big\} \\
& - \sum_{i=1}^m \frac{x_3 x_1^{(i-1)}\gamma}{(1-6\gamma)^i}\Big\{v(k+(\pm 3\ell_1,0,0,-(i-1)\ell_4,-(i-1)\ell_5)) \\
& \quad + v(k+(0,\pm 3\ell_2,0,-(i-1)\ell_4,-(i-1)\ell_5)) \\
& \qquad\qquad + v(k+(0,0,\pm 3\ell_3,-(i-1)\ell_4,-(i-1)\ell_5))\Big\},
\end{aligned} \tag{5.82}
$$

(b).
$$
\begin{aligned}
v(k) = {} & \frac{(1-6\gamma)^m}{{x_1}^m} v(k+(0,0,0,m\ell_4,m\ell_5)) \\
& - \sum_{i=1}^m \frac{x_2(1-6\gamma)^{i-1}}{{x_1}^i} v(k+(0,0,0,(i-2)\ell_4,(i-2)\ell_5)) \\
& - \sum_{i=1}^m \frac{x_3(1-6\gamma)^{i-1}}{{x_1}^i} v(k+(0,0,0,(i-3)\ell_4,(i-3)\ell_5)) \\
& - \sum_{i=1}^m \frac{\gamma(1-6\gamma)^{i-1}}{{x_1}^{(i-1)}}\Big[v(k+(\pm\ell_1,\ell_2,\ell_3,i\ell_4,i\ell_5)) \\
& + \sum_{i=1}^m \frac{x_3\gamma(1-6\gamma)^{(i-1)}}{x_1^i} v(k+(\pm 3\ell_1,\ell_2,\ell_3,i\ell_4,i\ell_5))\Big] \\
& \qquad + \sum_{i=1}^m \frac{x_2\gamma(1-6\gamma)^{(i-1)}}{x_1^i} v(k+(\pm 2\ell_1,\ell_2,\ell_3,i\ell_4,i\ell_5))\Big],
\end{aligned} \tag{5.83}
$$

$$
\begin{aligned}
\text{(c).}\qquad v(k) &= \frac{1}{\gamma^m} v(k + (-m\ell_1, 0, 0, -m\ell_4, -m\ell_5) \\
&+ \sum_{r=1}^{m} \frac{x_2}{x_1\gamma^i} v(k + (-i\ell_1, 0, 0, -(i+1)\ell_4, -(i+1)\ell_5)) \\
&+ \sum_{r=1}^{m} \frac{x_3}{x_1\gamma^i} v(k + (-i\ell_1, 0, 0, -(i+2)\ell_4, -(i+2)\ell_5)) \\
&- \sum_{i=1}^{m} \frac{1}{\gamma^{i-1}} \Big[v(k + (-(i+1)\ell_1, 0, 0, -(i-1)\ell_4, -(i-1)\ell_5)) \\
&\qquad + v(k + (-i\ell_1, \pm\ell_2, 0, -(i-1)\ell_4, -(i-1)\ell_5)) \\
&\qquad + v(k + (-i\ell_1, 0, \pm\ell_3, -(i-1)\ell_4, -(i-1)\ell_5)) \Big] \\
&- \sum_{i=1}^{m} \frac{x_2}{x_1\gamma^{i-1}} \Big[v(k + (-(i-2)\ell_1, 0, 0, -(i-1)\ell_4, -(i-1)\ell_5)) \\
&\qquad + v(k + (-(i+2)\ell_1, 0, 0, -(i-1)\ell_4, -(i-1)\ell_5)) \\
&\qquad + v(k + (-i\ell_1, \pm 2\ell_2, 0, -(i-1)\ell_4, -(i-1)\ell_5)) \\
&\qquad + v(k + (-i\ell_1, 0, \pm 2\ell_3, -(i-1)\ell_4, -(i-1)\ell_5)) \Big] \\
&- \sum_{i=1}^{m} \frac{x_3}{x_1\gamma^{i-1}} \Big[v(k + (-(i-3)\ell_1, 0, 0, -(i-1)\ell_4, -(i-1)\ell_5)) \\
&\qquad + v(k + (-(i+3)\ell_1, 0, 0, -(i-1)\ell_4, -(i-1)\ell_5)) \\
&\qquad + v(k + (-i\ell_1, \pm 3\ell_2, 0, -(i-1)\ell_4, -(i-1)\ell_5)) \\
&\qquad + v(k + (-i\ell_1, 0, \pm 3\ell_3, -(i-1)\ell_4, -(i-1)\ell_5)) \Big] \\
&- \sum_{i=1`}^{m} \frac{(1-6\gamma)}{x_1\gamma^i} v(k + (-i\ell_1, 0, 0, -(i-1)\ell_4, -(i-1)\ell_5)), \qquad (5.84)
\end{aligned}
$$

$$
\begin{aligned}
\text{(d).}\qquad v(k) &= \frac{1}{\gamma^m} v(k + (m\ell_1, 0, 0, -m\ell_4, -m\ell_5)) \\
&+ \sum_{r=1}^{m} \frac{x_2}{x_1\gamma^i} v(k(+i\ell_1, 0, 0, -(i+1)\ell_4, -(i+1)\ell_5)) \\
&+ \sum_{r=1}^{m} \frac{x_3}{x_1\gamma^i} v(k(+i\ell_1, 0, 0, -(i+2)\ell_4, -(i+2)\ell_5)) \\
&- \sum_{i=1}^{m} \frac{1}{\gamma^{i-1}} \Big[v(k + ((i+1)\ell_1, 0, 0, -(i-1)\ell_4, -(i-1)\ell_5))
\end{aligned}
$$

$$
\begin{aligned}
&+v(k+(i\ell_1,\pm\ell_2,0,-(i-1)\ell_4,-(i-1)\ell_5))\\
&+v(k+(i\ell_1,0,\pm\ell_3,-(i-1)\ell_4,-(i-1)\ell_5))\Big]\\
-\sum_{i=1}^{m}\frac{x_2}{x_1\gamma^{i-1}}\Big[&v(k+((i-2)\ell_1,0,0,-(i-1)\ell_4,-(i-1)\ell_5))\\
&+v(k+((i+2)\ell_1,0,0,-(i-1)\ell_4,-(i-1)\ell_5))\\
&+v(k+(i\ell_1,\pm 2\ell_2,0,-(i-1)\ell_4,-(i-1)\ell_5))\\
&+v(k+(i\ell_1,0,\pm 2\ell_3,-(i-1)\ell_4,-(i-1)\ell_5))\Big]\\
-\sum_{i=1}^{m}\frac{x_3}{x_1\gamma^{i-1}}\Big[&v(k+((i-3)\ell_1,0,0,-(i-1)\ell_4,-(i-1)\ell_5))\\
&+v(k+((i+3)\ell_1,0,0,-(i-1)\ell_4,-(i-1)\ell_5))\\
&+v(k+(i\ell_1,\pm 3\ell_2,0,-(i-1)\ell_4,-(i-1)\ell_5))\\
&+v(k+(i\ell_1,0,\pm 3\ell_3,-(i-1)\ell_4,-(i-1)\ell_5))\Big]\\
-\sum_{i=1}^{m}&\frac{(1-6\gamma)}{x_1\gamma^{i}}v(k+(i\ell_1,0,0,-(i-1)\ell_4,-(i-1)\ell_5)). \qquad (5.85)
\end{aligned}
$$

Proof. The proof of this theorem is similar to the proof of Theorem 5.2.2. □

Theorem 5.4.15. *Consider (5.67) and denote* $v(k_1\pm\ell_1,*)=v(k_1+\ell_1,*)+v(k_1-\ell_1,*), v(k_1\pm 2\ell_1,*)=v(k_1+2\ell_1,*)+v(k_1-2\ell_1,*)$ *and* $v(k_1\pm 3\ell_1,*)=v(k_1+3\ell_1,*)+v(k_1-3\ell_1,*)$*. Then, the following four types of solutions of the equation (5.67) are equivalent: If* $\underset{x(\ell)}{\Delta}v(k)=v(k)-x_1v(k-\ell)-x_2v(k-2\ell)-...-x_nv(k-n\ell)$*, then the corresponding equation (5.68) has a solution of the form*

$$
\begin{aligned}
\text{(a). } v(k)=&\frac{{x_1}^n}{(1-6\gamma)^n}v(k-(0,0,0,n\ell_4,n\ell_5))\\
&-\sum_{s=1}^{n}\frac{\gamma {x_1}^s}{(1-6\gamma)^s}\Big\{v(k-(\pm\ell_1,0,0,(s-1)\ell_4,(s-1)\ell_5))
\end{aligned}
$$

$$+v(k-(0,\pm\ell_2,0,(s-1)\ell_4,(s-1)\ell_5))+v(k-(0,0,\pm\ell_3,(s-1)\ell_4),(s-1)\ell_5))\Big\}$$

$$+\sum_{r=2}^{m}\Big\{\sum_{s=1}^{n}\frac{x_r{x_1}^{s-1}}{(1-6\gamma)^s}\Big[v(k-(0,0,0,-(s+(r-1))\ell_4,(s+(r-1))\ell_5))$$

$$-\gamma v(k-(\pm r\ell_1,0,0,(s-1)\ell_4,(s-1)\ell_5))+v(k-(0,\pm r\ell_2,0,(s-1)\ell_4,(s-1)\ell_5))$$

$$+v(k-(0,0,\pm r\ell_3,(s-1)\ell_4,(s-1)\ell_5))\Big]\Big\}, \tag{5.86}$$

(b). $$v(k)=\frac{(1-6\gamma)^n}{{x_1}^n}v(k+(0,0,0,n\ell_4,n\ell_5))$$
$$-\sum_{s=1}^{n}\frac{\gamma(1-6\gamma)^{s-1}}{{x_1}^{s-1}}\Big[v(k+(\pm\ell_1,0,0,+s\ell_4,+s\ell_5))$$
$$+v(k+(0,\pm\ell_2,0,+s\ell_4,+s\ell_5))+v(k+(0,0,\pm\ell_3,s\ell_4,s\ell_5))\Big]$$
$$-\sum_{r=2}^{m}\Big\{\sum_{s=1}^{n}\frac{x_r(1-6\gamma)^{s-1}}{{x_1}^s}\Big[v(k+(0,0,0,(s-r)\ell_4,(s-r)\ell_5))$$
$$-\gamma v(k+(\pm r\ell_1,0,0,+s\ell_4,+s\ell_5))+v(k+(0,\pm r\ell_2,0,s\ell_4,+s\ell_5))$$
$$+v(k+(0,0,\pm r\ell_3,s\ell_4,s\ell_5))\Big]\Big\}, \tag{5.87}$$

(c). $$v(k)=\frac{1}{\gamma^n}v(k+(-n\ell_1,0,0,-n\ell_4,-n\ell_5))$$
$$-\sum_{s=0}^{n}\frac{1}{\gamma^{s-1}}\Big[v(k+(-(s+1)\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5))$$
$$+v(k+(-s\ell_1,\pm\ell_2,0,-(s-1)\ell_4,-(s-1)\ell_5))$$
$$+v(k+(-s\ell_1,0,\pm\ell_3,-(s-1)\ell_4,-(s-1)\ell_5))\Big]$$
$$-\sum_{s=0}^{n}\frac{(1-6\gamma)}{x_1\gamma^s}v(k+(-s\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5))$$
$$-\sum_{r=2}^{m}\Big\{\sum_{s=1}^{n}\frac{x_r}{x_1\gamma^{s-1}}\Big[v(k+(-(s-r)\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5))\Big]$$
$$+v(k+(-(s+r)\ell_1,0,0,-(s-1)\ell_4,-(s-1)\ell_5))$$
$$+v(k+(-s\ell_1\pm r\ell_2,0,-(s-1)\ell_4,-(s-1)\ell_5))$$

$$
\begin{aligned}
&\quad + v(k + (-s\ell_1 \pm, 0, r\ell_3, -(s-1)\ell_4, -(s-1)\ell_5))\Big\}\\
&+ \sum_{r=2}^{m} \left\{ \sum_{s=1}^{n} \frac{x_r}{x_1 \gamma^s} v(k + (-s\ell_1, 0, 0, -(s+(r-1))\ell_4, -(s+(r-1))\ell_5)) \right\},
\end{aligned}
$$

$$
\begin{aligned}
\text{(d). } v(k) &= \frac{1}{\gamma^n} v(k + (n\ell_1, 0, 0, -n\ell_4, -n\ell_5))\\
&- \sum_{s=0}^{n} \frac{1}{\gamma^{s-1}} \Big[v(k(+(s+1)\ell_1, 0, 0, -(s-1)\ell_4, -(s-1)\ell_5))\\
&+ v(k(+s\ell_1, \pm\ell_2, 0, -(s-1)\ell_4, -(s-1)\ell_5))\\
&+ v(k(+s\ell_1, 0, \pm\ell_3, -(s-1)\ell_4, -(s-1)\ell_5))\Big]\\
&- \sum_{s=0}^{n} \frac{(1-6\gamma)}{x_1 \gamma^s} v(k(+s\ell_1, 0, 0, -(s-1)\ell_4, -(s-1)\ell_5))\\
&- \sum_{r=2}^{m} \Bigg\{ \sum_{s=1}^{n} \frac{x_r}{x_1 \gamma^{s-1}} \Big[v(k(+(s-r)\ell_1, 0, 0, -(s-1)\ell_4, -(s-1)\ell_5)) \Big]\\
&+ v(k + ((s+r)\ell_1, 0, 0, -(s-1)\ell_4, -(s-1)\ell_5))\\
&+ v(k + (s\ell_1 \pm r\ell_2, 0, -(s-1)\ell_4, -(s-1)\ell_5))\\
&+ v(k + (s\ell_1 \pm, 0, r\ell_3, -(s-1)\ell_4, -(s-1)\ell_5))\Bigg\}\\
&+ \sum_{r=2}^{m} \left\{ \sum_{s=1}^{n} \frac{x_r}{x_1 \gamma^s} v(k + (s\ell_1, 0, 0, -(s+(r-1))\ell_4, -(s+(r-1))\ell_5)) \right\}.
\end{aligned}
$$

$$
+ v(k_1 - (i+r)\ell_1, k_2, k_3 - (i-1)\ell_3) + v(k_1 - i\ell_1, k_2 \pm r\ell_2, k_3 - (i-1)\ell_3)\Big\}. \quad (5.88)
$$

From our derivations and findings, we conclude that in order to introduce several factors in the heat flow, we have to select proper value for m and $x = (x_1, x_2, ..., x_m)$ depending on the climate of heat flows which will give exact solutions.

Chapter 6

IVP of Heat Flows in Non-homogeneous Materials

6.1 Introduction

This chapter provides solutions of the initial value problem of heat flows of non-homogeneous materials. The methodology of induction is employed to guide us reach the destination. With Newton's law of cooling as the basis, the equation for heat transfer of the rod made of four different materials is formulated as the preliminary case. The solution arrived at for the above problem is generalized for the case of the rod with multiple materials. The results put forth in this book work are validated by numerical examples.

6.2 IVP of non-homogeneous materials

Consider a rod of non-homogenous materials with parameters k_1, k_2, ℓ_1 and ℓ_2. Let ℓ_1, ℓ_2 be the shift values of k_1 and k_2 respectively. Let $v(j_1, j_2)$ be the temperature at the initial position j_1 and at initial time j_2 and $\alpha(k_1)$ be the heat conductivity of rod at k_1. Non-homogeneous materials are introduced by $\alpha(k)$. Since the rod is non-homogeneous materials, the corresponding heat equation can be expressed as

$$v(k_1, k_2) = \alpha(k_1-\ell_1)v(k_1-\ell_1, k_2-\ell_2)+\alpha(k_1)v(k_1, k_2-\ell_2)+\alpha(k_1+\ell_1)v(k_1\pm\ell_1, k_2-\ell_2). \tag{6.1}$$

The equation (6.1) can be rewritten as

$$v(k_1, k_2) = \alpha(k_1)v(k_1, k_2 - \ell_2) + \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - \ell_2). \tag{6.2}$$

Theorem 6.2.1. *With usual notations, (6.2) has a solution of the form*

$$v(k_1, k_2) = \alpha^n(k_1)v(k_1, j_2) + \sum_{s=0}^{n-1} \alpha^{n-1-s}\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + s\ell_2). \tag{6.3}$$

Proof. Replacing k_2 by $k_2 - \ell_2$ in (6.2) gives

$$v(k_1, k_2 - \ell_2) = \alpha(k_1)v(k_1, k_2 - 2\ell_2) + \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - 2\ell_2).$$

From (6.2),

$$v(k_1, k_2) = \alpha^2(k_1)v(k_1, k_2-2\ell_2)+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1, k_2-2\ell_2)+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1, k_2-\ell_2). \tag{6.4}$$

Replacing k_2 by $k_2 - 2\ell_2$ in (6.4) gives

$$v(k_1, k_2 - 2\ell_2) = \alpha(k_1)v(k_1, k_2 - 3\ell_2) + \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - 3\ell_2)$$

$$v(k_1,k_2)=\alpha^3(k_1)v(k_1,k_2-3\ell_2)+\alpha^2(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-3\ell_2)$$
$$+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-2\ell_2)+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-\ell_2). \tag{6.5}$$

In general, we arrive at

$$v(k_1,k_2)=\alpha^n(k_1)v(k_1,k_2-n\ell_2)+\sum_{q=1}^{n}\alpha^{q-1}\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-q\ell_2).$$

By taking $k_2-n\ell_2=j_2$, the above relation becomes

$$v(k_1,k_2)=\alpha^n(k_1)v(k_1,j_2)+\sum_{q=1}^{n}\alpha^{q-1}\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2+(n-q)\ell_2),$$

by taking $n-s=q$, which is same as the solution. □

The following example illustrates of Theorem 6.2.1.

Example 6.2.2. *Consider (6.5), it can be expressed as*

$$v(k_1,k_2)=\alpha^n(k_1)v(k_1,j_2)+\sum_{s=1}^{n}\alpha^{n-s}\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2+(s-1)\ell_2), \tag{6.6}$$

where $k_2=j_2+n\ell_2$.

If $n=1$, *then* (6.3) *becomes* $v(k_1,k_2)=\alpha(k_1)v(k_1,j_2)+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2)$ *and*

$$v(k_1,j_2+\ell_2)=\alpha(k_1)v(k_1,j_2)+\left[\alpha(k_1+\ell_1)v(k_1+\ell_1,j_2)+\alpha(k_1-\ell_1)v(k_1-\ell_1,j_2)\right]. \tag{6.7}$$

If $n=2$, *then* (6.3) *becomes*

$$v(k_1,k_2)=\alpha^2(k_1)v(k_1,j_2)+\alpha(k_1)\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2)+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2+\ell_2)$$

and hence we have $v(k_1,j_2+2\ell_2)=\alpha^2(k_1)v(k_1,j_2)$

$$+\alpha(k_1)\left[\alpha(k_1+\ell_1)v(k_1+\ell_1,j_2)+\alpha(k_1-\ell_1)v(k_1-\ell_1,j_2)\right]$$
$$+\left[\alpha(k_1+\ell_1)v(k_1+\ell_1,j_2+\ell_2)+\alpha(k_1-\ell_1)v(k_1-\ell_1,j_2+\ell_2)\right]. \tag{6.8}$$

Similarly if $n = 3$, we can have

$v(k_1, k_2) = \alpha^3(k_1)v(k_1, j_2) + \alpha^2(k_1)\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2)$

$+ \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + \ell_2) + \alpha(k_1 \pm \ell_1, j_2 + 2\ell_2)$, *which gives*

$$\begin{aligned} v(k_1, j_2 + 3\ell_2) = {} & \alpha^3(k_1)v(k_1, j_2) \\ & + \alpha^2(k_1)\left[\alpha(k_1 + \ell_1)v(k_1 + \ell_1, j_2) + \alpha(k_1 - \ell_1)v(k_1 - \ell_1, j_2)\right] \\ & + \alpha(k_1)\left[\alpha(k_1 + \ell_1)v(k_1 + \ell_1, j_2 + \ell_2) + \alpha(k_1 - \ell_1)v(k_1 - \ell_1, j_2 + \ell_2)\right] \\ & + \left[\alpha(k_1 + \ell_1)v(k_1 + \ell_1, j_2 + 2\ell_2) + \alpha(k_1 - \ell_1)v(k_1 - \ell_1, j_2 + 2\ell_2)\right]. \end{aligned} \tag{6.9}$$

Example 6.2.3. *To find the conductivity rate $\alpha(k_1), \alpha(k_1 + \ell_1), \alpha(k_1 - \ell_1)$, for example consider the initial values given at $k_1, k_1 + \ell_1, k_1 - \ell_1$ as given below:*

$v(k_1, j_2) = 1$, $v(k_1 + \ell_1, j_2) = 2$, $v(k_1 - \ell_1, j_2) = 3$, $v(k_1 + \ell_1, j_2 + \ell_2) = 4$

$v(k_1 - \ell_1, j_2 + \ell_2) = 5$, $v(k_1 + \ell_1, j_2 + 2\ell_2) = 5.5$, $v(k_1 - \ell_1, j_2 + 2\ell_2) = 6$

$v(k_1, j_2 + \ell_2) = 2$, $v(k_1, j_2 + 2\ell_2) = 3$ *and* $v(k_1, j_2 + 3\ell_2) = 4$.

Let, $\alpha(k_1) = x_0$, $\alpha(k_1 + \ell_1) = x_1$, $\alpha(k_1 - \ell_1) = x_2$

$$(6.7) \Rightarrow x_0(1) + [x_1(2) + x_2(3)] = 2 \Rightarrow x_0 + 2x_1 + 3x_2 = 2. \tag{6.10}$$

$$\begin{aligned} (6.8) \Rightarrow {} & x_0^2(1) + x_0\left[x_1(2) + x_2(3)\right] + \left[x_1(4) + x_2(5)\right] = 3 \\ \Rightarrow {} & x_0^2 + 2x_0x_1 + 3x_0x_2 + 4x_1 + 5x_2 = 3 \\ \Rightarrow {} & x_0\left[x_0 + 2x_1 + 3x_2\right] + 4x_1 + 5x_2 = 3 \Rightarrow 2x_0 + 4x_1 + 5x_2 = 3. \end{aligned} \tag{6.11}$$

$$\begin{aligned} (6.9) \Rightarrow {} & x_0^3(1) + x_0^2\left[x_1(2) + x_2(3)\right] + x_0\left[x_1(4) + x_2(5)\right] + \left[x_1(5.5) + x_2(6)\right] = 4 \\ \Rightarrow {} & x_0^3 + 2x_0^2x_1 + 3x_0^2x_2 + 4x_0x_1 + 5x_0x_2 + 5.5x_1 + 6x_2 = 4 \end{aligned}$$

$$\Rightarrow x_0\left[x_0^2+2x_0x_1+3x_0x_2+4x_1+5x_2\right]+5.5x_1+6x_2=4$$

$$\Rightarrow 3x_0+5.5x_1+6x_2=4. \tag{6.12}$$

Solving the above equations, we get $x_2=1$*,* $x_1=-2$*,* $x_0=3$*.*

Example 6.2.4. *Consider the parameters in heat flows, let* $v(k_1,j_2)$ *be the initial temperature at the initial time* j_2 *at position* k_1 *of a long rod.*

(i.e) $v(j_1,j_2), v(j_1+\ell_1,j_2), v(j_1+2\ell_1,j_2), v(j_1+3\ell_1,j_2), ..., v(j_1+p\ell_1,j_2)$ *be the temperature at initial time* j_2*.*

Let us assume that initial values be

$v(j_1,j_2)=v(j_1+\ell_1,j_2)=v(j_1+2\ell_1,j_2)=...=v(j_1+p\ell_1,j_2)=1.$

If $n=4$*, we arrive at*

$$\begin{aligned} v(k_1,j_2+4\ell_2)=&\ \alpha^4(k_1)v(k_1,j_2)+\sum_{s=1}^{4}\alpha^{n-s}(k_1)\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2+(s-1)\ell_2)\\ =&\alpha^4(k_1)v(k_1,j_2)+\alpha^3(k_1)\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2)\\ &+\alpha^2(k_1)\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2+\ell_2)+\alpha(k_1)\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2+2\ell_2)\\ &+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2+3\ell_2). \end{aligned}$$

As, we assume j_1 *as initial values, then*

$$\begin{aligned} v(j_1,j_2+4\ell_2)=&\ \alpha^4(j_1)v(j_1,j_2)\\ &+\alpha^3(j_1)\left[\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2)+\alpha(j_1-\ell_1)v(j_1-\ell_1,j_2)\right]\\ &+\alpha^2(j_1)\left[\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2+\ell_2)+\alpha(j_1-\ell_1)v(j_1-\ell_1,j_2+\ell_2)\right]\\ &+\alpha(j_1)\left[\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2+2\ell_2)+\alpha(j_1-\ell_1)v(j_1-\ell_1,j_2+2\ell_2)\right]\\ &+\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2+3\ell_2)+\alpha(j_1-\ell_1)v(j_1-\ell_1,j_2+3\ell_2). \end{aligned} \tag{6.13}$$

From (6.7), we find

$$v(j_1, j_2+\ell_2) = 0 + \alpha(j_i)v(j_1, j_2) + \alpha(j_1+\ell_1)v(j_1+\ell_1, j_2)$$

$$= x_0(1) + x_1(1) = x_0 + x_1 = 3 - 2 = 1.$$

(i). $v(j_1+\ell_1, j_2+\ell_2) = \alpha(j_1)v(j_1, j_2) + \alpha(j_1+\ell_1)v(j_1+\ell_1, j_2) + \alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2)$

$$= x_0 + x_1 + x_2 = 3 - 2 + 1 = 2.$$

(ii). $v(j_1-\ell_1, j_2+\ell_2) = \alpha(j_1-2\ell_1)v(j_1-2\ell_1, j_2) + \alpha(j_1-\ell_1)v(j_1-\ell_1, j_2) + \alpha(j_1)v(j_1, j_2)$

$$= 0 + 0 + x_0 = 3.$$

(iii). $v(j_1+\ell_1, j_2+2\ell_2) = \alpha(j_1)v(j_1, j_2+\ell_2) + \alpha(j_1+\ell_1)v(j_1+\ell_1, j_2+\ell_2)$

$$+ \alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2+\ell_2)$$

$$v(j_1+2\ell_1, j_2+\ell_2) = \alpha(j_1+\ell_1)v(j_1+\ell_1, j_2) + \alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2)$$

$$+ \alpha(j_1+3\ell_1)v(j_1+3\ell_1, j_2)$$

$$= x_1 + x_2 + x_0 = 3 - 2 + 1 = 2$$

$$= x_0(1) + x_1(2) + x_2(2) = 3 - 4 + 2 = 1.$$

(iv). $v(j_1-\ell_1, j_2+2\ell_2) = 0 + 0 + \alpha(j_1)v(j_1, j_2+\ell_2) = x_0(1) = 3.$

(v). $v(j_1+\ell_1, j_2+3\ell_2) = \alpha(j_1)v(j_1, j_2+2\ell_2) + \alpha(j_1+\ell_1)v(j_1+\ell_1, j_2+2\ell_2)$

$$+ \alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2+2\ell_2)$$

$$v(j_1, j_2+2\ell_2) = 0 + \alpha(j_1)v(j_1, j_2+\ell_2) + \alpha(j_1+\ell_1)v(j_1+\ell_1, j_2+\ell_2)$$

$$= x_0(1) + x_1(2) = 3 - 4 = -1$$

$$v(j_1+2\ell_1, j_2+2\ell_2) = \alpha(j_1+\ell_1)v(j_1+\ell_1, j_2+\ell_2) + \alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2\ell_2)$$

$$+\alpha(j_1+3\ell_1)v(j_1+3\ell_1, j_2+\ell_2)$$

(a). $v(j_1+3\ell_1, j_2+\ell_2) = \alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2) + \alpha(j_1+3\ell_1)v(j_1+3\ell_1, j_2)$

$$+\alpha(j_1+4\ell_1)v(j_1+4\ell_1,j_2)$$

$$= x_2(1)+x_0(1)+x_1(1) = 1+3-2 = 2$$

$$v(j_1+2\ell_1,j_2+\ell_2) = x_1(2)+x_2(2)+x_0(2) = (-2)(2)+(1)(2)+(3)(2) = -4+2+6 = 4$$

$$v(j_1+\ell_1,j_2+3\ell_2) = x_0(-1)+x_1(1)+x_2(4) = -3-2+4 = -1.$$

(vi). $v(j_1-\ell_1,j_2+3\ell_2) = 0+0+\alpha(j_1)v(j_1,j_2+2\ell_2)$

$$= x_0(-1) = 3(-1) = -3.$$

(6.13) $\Rightarrow v(j_1,j_2+4\ell_2) = x_0^4(1)+x_0^3\{x_1(1)+x_2(1)\}+x_0^2\{x_1(2)+x_2(3)\}$

$$+x_0\{x_1(1)+x_2(3)\}+x_1(-1)+x_2(-3)$$

$$= 3^4+3^3(-2+1)+3^2(-4+3)+3(-2+3)+2-3 = 47$$

Similarly, we are able to find the heat temperature $v(j_1,j_2+q\ell_2)$ of the given material by using the temperature at initial time j_2 with the three conductivity rate $\alpha(k_1),\alpha(k_1+\ell_1)$ and $\alpha(k_1-\ell_1)$.

Example 6.2.5. *Consider the equation (6.13), to find $v(j_1+p\ell_1,j_2)$:*

(6.13) $\Rightarrow v(j_1,j_2+4\ell_2) = \alpha^4(j_1)v(j_1,j_2)$

$$+\alpha^3(j_1)\left[\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2)+\alpha(j_1-\ell_1)v(j_1-\ell_1,j_2)\right]$$

$$+\alpha^2(j_1)\left[\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2+\ell_2)+\alpha(j_1-\ell_1)v(j_1-\ell_1,j_2+\ell_2)\right]$$

$$+\alpha(j_1)\left[\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2+2\ell_2)+\alpha(j_1-\ell_1)v(j_1-\ell_1,j_2+2\ell_2)\right]$$

$$+\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2+3\ell_2)+\alpha(j_1-\ell_1)v(j_1-\ell_1,j_2+3\ell_2)$$

Now replacing j_1 by $j_1+\ell_1$, we find

$$v(j_1+\ell_1,j_2+4\ell_2) = \alpha^4(j_1+\ell_1)v(j_1+\ell_1,j_2)$$

$$+\alpha^3(j_1+\ell_1)\left[\alpha(j_1+2\ell_1)v(j_1+2\ell_1,j_2)+\alpha(j_1)v(j_1,j_2)\right]$$

$$+\alpha^2(j_1+\ell_1)\left[\alpha(j_1+2\ell_1)v(j_1+2\ell_1,j_2+\ell_2)+\alpha(j_1)v(j_1,j_2+\ell_2)\right]$$

$$+\alpha(j_1+\ell_1)\left[\alpha(j_1+2\ell_1)v(j_1+2\ell_1,j_2+2\ell_2)+\alpha(j_1)v(j_1,j_2+2\ell_2)\right]$$

$$+\alpha(j_1+2\ell_1)v(j_1+2\ell_1,j_2+3\ell_2)+\alpha(j_1)v(j_1,j_2+3\ell_2) \tag{6.14}$$

$$=x_1^4(1)+x_1^3(x_2(1)+x_0(1))+x_1^2(x_2(2)+x_0(1))$$

$$+x_1(x_2(4)+x_0(-1))+x_2(14)+x_0(-5)$$

$$=16+(-8)(1+3)+4(2+3)+(-2)(4-3)+14(1)-15=1.$$

Replacing j_1 *by* $j_1+2\ell_2$ *in* (6.14) *yields*

$$v(j_1+2\ell_1,j_2+4\ell_2)=\alpha^4(j_1+2\ell_1)v(j_1+2\ell_1,j_2)$$

$$+\alpha^3(j_1+2\ell_1)\left[\alpha(j_1+3\ell_1)v(j_1+3\ell_1,j_2)+\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2)\right]$$

$$+\alpha^2(j_1+2\ell_1)\left[\alpha(j_1+3\ell_1)v(j_1+3\ell_1,j_2+\ell_2)+\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2+\ell_2)\right]$$

$$+\alpha(j_1+2\ell_1)\left[\alpha(j_1+3\ell_1)v(j_1+3\ell_1,j_2+2\ell_2)+\alpha(j_1+\ell_1)v(j_1+\ell_1,j_2+2\ell_2)\right]$$

$$+\alpha(j_1+2\ell_1)v(j_1+2\ell_1,j_2+3\ell_2)+\alpha(j_1)v(j_1,j_2+3\ell_2)$$

(i). $v(j_1+3\ell_1,j_2+3\ell_2)=\alpha(j_1+2\ell_1)v(j_1+2\ell_1,j_2+2\ell_2)$

$$+\alpha(j_1+3\ell_1)v(j_1+3\ell_1,j_2+2\ell_2)+\alpha(j_1+4\ell_1)v(j_1+4\ell_1,j_2+2\ell_2)$$

$$v(j_1+4\ell_1,j_2+2\ell_2)=\alpha(j_1+3\ell_1)v(j_1+3\ell_1,j_2+\ell_2)$$

$$+\alpha(j_1+4\ell_1)v(j_1+4\ell_1,j_2+\ell_2)+\alpha(j_1+5\ell_1)v(j_1+5\ell_1,j_2+\ell_2)$$

$$v(j_1+5\ell_1,j_2+\ell_2)=\alpha(j_1+4\ell_1)v(j_1+4\ell_1,j_2)+\alpha(j_1+5\ell_1)v(j_1+5\ell_1,j_2)$$

$$+\alpha(j_1+6\ell_1)v(j_1+6\ell_1,j_2)=2$$

$$v(j_1+4\ell_1,j_2+2\ell_2)=x_0(2)+x_1(2)+x_2(2)=4$$

$v(j_1+3\ell_1, j_2+3\ell_2) = x_2(4)+x_0(4)+x_1(4) = 8$

$$v(j_1+2\ell_1, j_2+4\ell_2) = x_2^4(1)+x_2^3(x_0(1)+x_1(1))+x_2^2(x_0(2)+x_1(2))$$

$$+x_2(x_0(4)+x_1(1))+x_0(8)+x_1(-1)$$

$$=1+1(3-2)+1(6+2-2)+(12-3)+24+2=40.$$

Replacing j_1 by $j_1+3\ell_2$ in (6.14) gives

$v(j_1+3\ell_1, j_2+4\ell_2) = \alpha^4(j_1+3\ell_1)v(j_1+3\ell_1, j_2)$

$+\alpha^3(j_1+3\ell_1)\left[\alpha(j_1+4\ell_1)v(j_1+4\ell_1, j_2)+\alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2)\right]$

$+\alpha^2(j_1+3\ell_1)\left[\alpha(j_1+4\ell_1)v(j_1+4\ell_1, j_2+\ell_2)+\alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2+\ell_2)\right]$

$+\alpha(j_1+3\ell_1)\left[\alpha(j_1+4\ell_1)v(j_1+4\ell_1, j_2+2\ell_2)+\alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2+2\ell_2)\right]$

$+\alpha(j_1+4\ell_1)v(j_1+4\ell_1, j_2+3\ell_2)+\alpha(j_1+2\ell_1)v(j_1+2\ell_1, j_2+3\ell_2)$

(i). $v(j_1+4\ell_1, j_2+3\ell_2) = \alpha(j_1+3\ell_1)v(j_1+3\ell_1, j_2+2\ell_2)$

$+\alpha(j_1+4\ell_1)v(j_1+4\ell_1, j_2+2\ell_2)+\alpha(j_1+5\ell_1)v(j_1+5\ell_1, j_2+2\ell_2)$

$v(j_1+5\ell_1, j_2+2\ell_2) = \alpha(j_1+4\ell_1)v(j_1+4\ell_1, j_2+\ell_2)$

$+\alpha(j_1+5\ell_1)v(j_1+5\ell_1, j_2+\ell_2)+\alpha(j_1+6\ell_1)v(j_1+6\ell_1, j_2+\ell_2)$

$$v(j_1+6\ell_1, j_2+\ell_2) = \alpha(j_1+5\ell_1)v(j_1+5\ell_1, j_2)+\alpha(j_1+6\ell_1)v(j_1+6\ell_1, j_2)$$

$$+\alpha(j_1+7\ell_1)v(j_1+7\ell_1, j_2) = 2$$

$v(j_1+5\ell_1, j_2+2\ell_2) = x_1(2)+x_2(2)+x_0(2) = 4$

$v(j_1+4\ell_1, j_2+3\ell_2) = x_2(4)+x_0(4)+x_1(4) = 8$

$$v(j_1+3\ell_1, j_2+4\ell_2) = x_0^4(1)+x_0^3(x_1(1)+x_2(1))+x_0^2(x_1(2)+x_2(2))$$

$$+x_0(x_1(4)+x_2(1))+x_1(4)+x_2(14) = 30.$$

In the similar way, we are able to find the temperature $v(j_1+p\ell_1, j_2+q\ell_2)$.

6.3 IVP of rod made of four non-homogeneous materials

Consider a rod of non homogenous materials with parameters k_1, k_2, ℓ_1 and ℓ_2. Let ℓ_1, ℓ_2 be the shift values of k_1 and k_2 respectively. Let $v(j_1, j_2)$ be the temperature at the initial position j_1 and at initial time j_2 and $\alpha(k_1)$ be the heat conductivity of rod at k_1. Since the rod is non-homogeneous materials, the corresponding heat equation can be expressed as

$$v(k_1, k_2) = \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, k_2 - \ell_2) + \alpha(k_1 - \ell_1)v(k_1 - \ell_1, k_2 - \ell_2)$$
$$+\alpha(k_1)v(k_1, k_2 - \ell_2) + \alpha(k_1 + \ell_1)v(k_1 + \ell_1, k_2 - \ell_2),$$

which can be expressed as

$$v(k_1, k_2) = \alpha(k_1)v(k_1, k_2 - \ell_2) + \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - \ell_2)$$
$$+ \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, k_2 - \ell_2). \tag{6.15}$$

Theorem 6.3.1. *With the usual notations, equation (6.15) has solution*

$$v(k_1, j_2 + n\ell_2) = \alpha^n(k_1)v(k_1, j_2)$$
$$+ \sum_{s=0}^{n-1} \alpha^{n-1-s}(k_1)\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + s\ell_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2 + s\ell_2)$$

(or) $v(k_1, k_2) = \alpha^n(k_1)v(k_1, j_2) + \sum_{s=1}^{n} \alpha^{n-s}(k_1)\{\alpha(k_1 \pm \ell_1)$

$$v(k_1 \pm \ell_1, j_2 + (s-1)\ell_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2 + (s-1)\ell_2)\}. \tag{6.16}$$

Proof. Replacing k_2 by $k_2 - \ell_2$ in (6.15) gives

$$v(k_1, k_2 - \ell_2) = \alpha(k_1)v(k_1, k_2 - 2\ell_2) + \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - 2\ell_2)$$
$$+\alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, k_2 - \ell_2).$$

From (6.15), we arrive

$$v(k_1,k_2)=\alpha^2(k_1)v(k_1,k_2-2\ell_2)+\alpha(k_1)\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-2\ell_2)$$
$$+\alpha(k_1)\alpha(k_1-2\ell_1)v(k_1-2\ell_1,k_2-2\ell_2)+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-\ell_2)$$
$$+\alpha(k_1-2\ell_1)v(k_1-2\ell_1,k_2-\ell_2). \tag{6.17}$$

Replacing k_2 by $k_2-2\ell_2$ in (6.17) gives

$$v(k_1,k_2-2\ell_2)=\alpha(k_1)v(k_1,k_2-3\ell_2)+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-3\ell_2)$$
$$+\alpha(k_1-2\ell_1)v(k_1-2\ell_1,k_2-3\ell_2)$$

$$v(k_1,k_2)=\alpha^3(k_1)v(k_1,k_2-3\ell_2)+\alpha^2(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-3\ell_2)$$
$$+\alpha^2(k_1)\alpha(k_1-2\ell_1)v(k_1-2\ell_1,k_2-3\ell_2)+\alpha(k_1)\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-2\ell_2)$$
$$+\alpha(k_1)\alpha(k_1-2\ell_1)v(k_1-2\ell_1,k_2-2\ell_2)+\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-\ell_2)$$
$$+\alpha(k_1-2\ell_1)v(k_1-2\ell_1,k_2-\ell_2).$$

In general, $v(k_1,k_2)=\alpha^n(k_1)v(k_1,k_2-n\ell_2)$

$$+\sum_{q=1}^{n}\alpha^{q-1}\left[\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-q\ell_2)+\alpha(k_1-2\ell_1)v(k_1-2\ell_1,k_2-q\ell_2)\right]. \tag{6.18}$$

By taking $k_2-n\ell_2=j_2$ in (6.18) becomes

$$v(k_1,k_2)=\alpha^n(k_1)v(k_1,j_2)$$
$$+\sum_{q=1}^{n}\alpha^{q-1}\{\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2+(n-q)\ell_2)+\alpha(k_1-2\ell_1)v(k_1-2\ell_1,j_2+(m-q)\ell_2)\}.$$

Taking $n-s=q$, which is same as above equation. □

Corollary 6.3.2. *Consider the particular cases for* $n = 1, 2$ *and* 3.

(i). If $n = 1, v(k_1, j_2 + \ell_2) = \alpha(k_1)v(k_1, j_2)$

$$+ [\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2)] . \tag{6.19}$$

(ii). If $n = 2, v(k_1, j_2 + 2\ell_2) = \alpha^2(k_1)v(k_1, j_2)$

$$+\alpha(k_1)\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2)\}$$

$$+ \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + \ell_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2 + \ell_2). \tag{6.20}$$

(iii). If $n = 3, v(k_1, j_2 + 3\ell_2) = \alpha^3(k_1)v(k_1, j_2),$

$+\alpha^2(k_1)\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2)\}$

$+ \alpha(k_1)\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + \ell_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2 + \ell_2)\}$

$$+ \{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + 2\ell_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2 + 2\ell_2)\}. \tag{6.21}$$

(iv). If $n = 4, v(k_1, j_2 + 3\ell_2) = \alpha^4(k_1)v(k_1, j_2)$

$+ \alpha^3(k_1)\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2)\}$

$+ \alpha^2(k_1)\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + \ell_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2 + \ell_2)\}$

$+ \alpha(k_1)\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + 2\ell_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2 + 2\ell_2)\}$

$$+ \{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + 3\ell_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2 + 3\ell_2)\}. \tag{6.22}$$

Example 6.3.3. *To explain how to find the conductivity rate* $\alpha(k_1), \alpha(k_1 + \ell_1)$, $\alpha(k_1 - \ell_1), \alpha(k_1 - 2\ell_1)$, *consider the assumed values given below at* $k_1, k_1 + \ell_1, k_1 - \ell_1$ $v(k_1, j_2) = 2,\ v(k_1 + \ell_1, j_2) = 1,\ v(k_1 - \ell_1, j_2) = 2,\ v(k_1 - 2\ell_1, j_2) = 4$

$v(k_1+\ell_1, j_2+\ell_2) = 3,\ v(k_1-\ell_1, j_2+\ell_2) = 2,\ v(k_1, j_2+\ell_2) = 4$

$v(k_1-2\ell_1, j_2+\ell_2) = 4,\ v(k_1+\ell_1, j_2+2\ell_2) = 4,\ v(k_1-\ell_1, j_2+2\ell_2) = 3$

$v(k_1, j_2+2\ell_2) = 5,\ v(k_1-2\ell_1, j_2+2\ell_2) = 6,\ v(k_1+\ell_1, j_2+3\ell_2) = 5$

$v(k_1-\ell_1, j_2+3\ell_2) = 6,\ v(k_1, j_2+3\ell_2) = 6,\ v(k_1-2\ell_1, j_2+3\ell_2) = 5.$

Let, $\alpha(k_1) = x_0,\ \alpha(k_1+\ell_1) = x_1,\ \alpha(k_1-\ell_1) = x_2,\ \alpha(k_1-2\ell_1) = x_3$

$$\Rightarrow 2x_0 + x_1 + 2x_2 + 4x_3 = 4. \tag{6.23}$$

(6.20) $\Rightarrow 2x_0^2 + x_0\,[x_1 + 2x_2 + 4x_3] + 3x_1 + 2x_2 + 4x_3 = 5,$

which gives $x_0\,[2x_0 + x_1 + 2x_2 + 4x_3] + 3x_1 + 2x_2 + 4x_3 = 5$

$$\Rightarrow 4x_0 + 3x_1 + 2x_2 + 4x_3 = 5. \tag{6.24}$$

(6.21) $\Rightarrow 2x_0^3 + x_0^2\,[x_1 + 2x_2 + 4x_3] + x_0\,[3x_1 + 2x_2 + 4x_3] + 4x_1 + 3x_2 + 6x_3 = 6$

which gives $x_0\,[2x_0^2 + x_0x_1 + 2x_0x_2 + 4x_0x_3 + 3x_1 + 2x_2 + 4x_3] + 4x_1 + 3x_2 + 6x_3 = 6$

$$\Rightarrow 5x_0 + 4x_1 + 3x_2 + 6x_3 = 6 \tag{6.25}$$

(6.22) $\Rightarrow 2x_0^4 + x_0^3\,[x_1 + 2x_2 + 4x_3] + x_0^2\,[3x_1 + 2x_2 + 4x_3] + x_0\,[4x_1 + 3x_2 + 6x_3]$

$$+ 5x_1 + 4x_2 + 5x_3 = 7$$

$x_0\,[2x_0^3 + x_0^2x_1 + 2x_0^2x_2 + 4x_0^2x_3 + 3x_1x_0 + 2x_2x_0 + 4x_3x_0 + 4x_1 + 3x_2 + 6x_3]$

$$+ 5x_1 + 4x_2 + 5x_3 = 7 \Rightarrow 6x_0 + 5x_1 + 4x_2 + 5x_3 = 7. \tag{6.26}$$

Solving the above equations, we get $x_0 = \dfrac{5}{2},\ x_1 = -2,\ x_2 = \dfrac{1}{2},\ x_3 = 0.$

Example 6.3.4. *Assume that $v(k_1, j_2)$ be the initial temperature at the initial time j_2 at position k_1. To understand the initial value problem, for example assume that initial values $v(j_1, j_2) = v(j_1 + \ell_1, j_2) = ... = v(j_1 + p\ell_1, j_2) = 1$.*

For the case $n = 5$, $v(k_1, k_2) = \alpha^n(k_1, j_2)$

$\sum_{s=1}^{n}\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + (s-1)\ell_2) + \alpha(k_1 - 2\ell_1)v(k_1 - 2\ell_1, j_2 + (s-1)\ell_2)\}$

$v(k_1, j_2 + 5\ell_2)$

(i). $v(j_1, j_2 + \ell_2) = x_0(2) + x_1(2) + x_2(2) + x_3(2) = 2.$

(ii). $v(j_1 + \ell_1, j_2 + \ell_2) = x_1(2) + x_2(2) + x_3(2) + x_0(2) = 2.$

(iii). $v(j_1 - \ell_1, j_2 + \ell_2) = x_2(2) + x_0(2) + x_3(2) + x_1(2) = 2.$

(iv). $v(j_1 - 2\ell_1, j_2 + \ell_2) = x_3(2) + x_2(2) + x_1(2) + x_0(2) = 2.$

(v).
$$\begin{aligned} v(j_1+\ell_1, j_2+2\ell_2) = {} & x_1^2(2)+x_1x_3(2)+x_1x_0(2)+x_1x_2(2)+x_3^2(2)+x_3x_2(2)+x_3x_1(2) \\ & +x_3x_0(2) + x_0(2) + x_2^2(2) + x_2x_0(2) + x_2x_3(2) + x_3x_1(2) \\ & =8 + 0 - 10 - 2 + 0 + 0 + 0 + 0 + 5 + \frac{1}{2} + \frac{5}{2} + 0 - 2 = 2. \end{aligned}$$

(vi). $v(j_1 - \ell_1, j_2 + 2\ell_2) = V(j_1 + \ell_1, j_2 + 2\ell_2) = 2.$

(vii).
$$\begin{aligned} v(j_1-2\ell_1, j_2+2\ell_2) = {} & x_3^2(2)+x_2x_3(2)+x_1x_3(2)+x_0x_3(2)+x_2(2)+x_1^2(2)+x_3x_1(2) \\ & +x_1x_0(2) + x_1x_2(2) + x_0^2(2) + x_1x_0(2) + x_2x_0(2) + x_3x_0(2) \\ & =9 - 12 - 10 + \frac{25}{2} + \frac{5}{2} = 2. \end{aligned}$$

(viii). $v(j_1 + \ell_1, j_2 + 3\ell_2)$

(a).
$$\begin{aligned} v(j_1, j_2 + 2\ell_2) = {} & x_0^2(2) + x_0x_1(2) + x_2x_0(2) + x_0x_3(2) + x_1(2) + x_2(2) + x_3(2) \\ & =1 + 0 - 4 + 0 - 10 + \frac{25}{2} + \frac{5}{2} + 0 - 2 = 2. \end{aligned}$$

(b). $v(j_1+2\ell_2, j_2+2\ell_2) = x_3^2(2)+x_3x_2(2)+x_3x_1(2)+x_0x_3(2)+x_2(2)+x_1(2)+x_0(2) = 2$

$$v(j_1+\ell_1, j_2+3\ell_3) = x_1^2(2) + x_1^2x_3(2) + x_0x_1^2(2) + x_1^2x_2(2) + x_1x_3(2) + x_1x_0(2)$$
$$+ x_1x_2(2) + x_3(2) + x_2(2) + x_0(2) = 2.$$

Finally, $v(j_1, j_2+5\ell_2) = x_0^5(2)+x_0^4x_1(2)+x_0^4x_2(2)+x_0^4x_3(2)+x_0^3[x_1(2)+x_2(2)+x_3(2)]$
$$+x_0^2[x_1(2)+x_2(2)+x_3(2)]+x_0[x_1(2)+x_2(2)+x_3(2)]+x_1(2)+x_2(2)+x_3(2) = 2.$$

6.4 IVP of heat flows of n-non-homogeneous materials

Since the rod is non-homogeneous materials, the corresponding heat equation of rod made of four materials can be expressed as

$$v(k_1,k_2) = \alpha(k_1)v(k_1,k_2-\ell_2) + \alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-\ell_2)$$
$$+ \alpha(k_1\pm 2\ell_1)v(k_1-2\ell_1,k_2-\ell_2). \tag{6.27}$$

Theorem 6.4.1. *The solution of initial value problem of heat equation of rod made of four non-homogeneous materials is given by*

$$v(k_1,k_2) = \alpha^n(k_1)v(k_1,j_2) + \sum_{s=1}^{n}\alpha^{n-s}(k_1)$$
$$\{\alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,j_2+(s-1)\ell_2) + \alpha(k_1\pm 2\ell_1)v(k_1\pm 2\ell_1,j_2+(s-1)\ell_2)\}. \tag{6.28}$$

Proof. Replacing k_2 by $k_2-\ell_2$ in (6.27) gives

$$v(k_1,k_2-\ell_2) = \alpha(k_1)v(k_1,k_2-2\ell_2) + \alpha(k_1\pm\ell_1)v(k_1\pm\ell_1,k_2-2\ell_2)$$
$$+\alpha(k_1\pm 2\ell_1)v(k_1-2\ell_1,k_2-\ell_2).$$

From (6.27), $v(k_1, k_2) = \alpha^2(k_1)v(k_1, k_2 - 2\ell_2) + \alpha(k_1)\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - 2\ell_2)$

$$+\alpha(k_1)\alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, k_2 - 2\ell_2) + \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - \ell_2)$$

$$+ \alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, k_2 - \ell_2). \tag{6.29}$$

Replacing k_2 by $k_2 - 2\ell_2$ in (6.29) gives

$v(k_1, k_2 - 2\ell_2) = \alpha(k_1)v(k_1, k_2 - 3\ell_2) + \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - 3\ell_2)$

$$+\alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, k_2 - 3\ell_2),$$

which yields $v(k_1, k_2) = \alpha^3(k_1)v(k_1, k_2 - 3\ell_2) + \alpha^2(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - 3\ell_2)$

$+\alpha^2(k_1)\alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, k_2 - 3\ell_2) + \alpha(k_1)\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - 2\ell_2)$

$+\alpha(k_1)\alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, k_2 - 2\ell_2) + \alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - \ell_2)$

$+\alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, k_2 - \ell_2)$.

In general, $v(k_1, k_2) = \alpha^n(k_1)v(k_1, k_2 - n\ell_2)$

$+ \sum_{q=1}^{n} \alpha^{q-1} \left[\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, k_2 - q\ell_2) + \alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, k_2 - q\ell_2)\right]$

By taking $k_2 - n\ell_2 = j_2$, we have

$v(k_1, k_2) = \alpha^n(k_1)v(k_1, j_2)$

$$+\sum_{q=1}^{n} \alpha^{q-1}\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + (n-q)\ell_2) + \alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, j_2 + (m-q)\ell_2)\}. \tag{6.30}$$

Taking $n - s = q$, which is same as above equation,

$v(k_1, j_2 + n\ell_2) = \alpha^n(k_1)v(k_1, j_2)$

$+ \sum_{s=0}^{n-1} \alpha^{n-1-s}(k_1)\{\alpha(k_1 \pm \ell_1)v(k_1 \pm \ell_1, j_2 + s\ell_2) + \alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, j_2 + s\ell_2)$. $\square$

Theorem 6.4.2. *The solution of initial value problem of heat equation of rod made of m non-homogeneous materials is given by*

$$v(k_1, k_2) = \alpha^n(k_1)v(k_1, j_2) + \sum_{t=1}^{m} \alpha^{n-s}(k_1)$$

$$\{\alpha(k_1 \pm t\ell_1)v(k_1 \pm t\ell_1, j_2 + (s-1)\ell_2) + \alpha(k_1 \pm 2\ell_1)v(k_1 \pm 2\ell_1, j_2 + (s-1)\ell_2)\}. \quad (6.31)$$

Proof. The proof is similar to the proof of Theorem 6.4.1. □

Hence, by knowing initial temperature at certain position of rod made of non-homogeneous materials it is possible to find the temperature at general position.

Chapter 7

Extorial Functions and its Applications

7.1 Introduction

This chapter presents the exact (closed form) solution of the heat equation. To achieve that, we need to introduce a new function denoted by $e_\nu(k_{(\ell)})$ entitled as extorial function which is obtained by replacing polynomials into polynomial factorials in the expansion of exponential function. Required identities involving difference operators with factorials are enumerated. And these identities are applied to obtain solutions of discrete partial difference equation for heat flow in the rod, thin plate, medium and rod made of multiple materials.

This chapter also focuses on the fractional difference operator which plays a pivotal role in studying numerous systems and has been widely applied in various areas of study [5, 15, 23, 24, 25, 26, 32, 33, 36, 37, 39]. Few to mention are population studies, economy price option and signal processing. Due to certain set backs caused by accumulative errors which in term makes long term and fast simulation difficult, less theoretical work are dedicated to fractional difference equations as well as the applications.

On the contrary to the above developments notable break-through has been achieved recently. A number of publications by Geodrica and Peterson on boundary value problems [18], Abu-saris and Cermark et.al., [3, 14], and D. Balenau et.al., [6, 40] in fractional difference equations have made the field popular and easily approachable. This chapter relates to formulation of the problem of current flow in the RL circuit into an discrete and fractional difference equation. The authors attempts to find an effective solution by applying extorial function.

7.2 Preliminaries

In this section, we present basic definitions of ℓ-difference operator and extorial functions. By Newton's law of cooling, the partial difference equation of the heat flow in the rod made of four material is controlled by the equation

$$v(k_1, k_2) = \sum_{r=-2}^{1} \alpha(k_1 + r\ell_1) v(k_1 + r\ell_1, k_2 - \ell_2). \tag{7.1}$$

Here, we show that the extorial function is a solution of the above heat equation. It also satisfies the heat equation for rod, thin plate and medium of homogeneous materials.

Definition 7.2.1. *For $-1 < \ell < 1, \ell \neq 0$ and $k, \nu \in (-\infty, \infty)$, and $m\nu + 1 \notin \{0, -1, -2, ...\}$, the ℓ-extorial function denoted as $e_\nu(k_\ell)$ is defined as*

$$e_\nu(k_\ell) = 1 + \frac{k_\ell^{(\nu)}}{\Gamma(\nu+1)} + \frac{k_\ell^{(2\nu)}}{\Gamma(2\nu+1)} + \frac{k_\ell^{(3\nu)}}{\Gamma(3\nu+1)} + \cdots + \infty. \tag{7.2}$$

If $\ell \in (-\infty, \infty), \ell \neq 0$ and k is a multiple of ℓ and $\nu \in N$, then (7.2) is defined, and which case all except finite terms of $e_\nu(k_\ell)$ are zero.

Definition 7.2.2. *[7] Let $\ell \neq 0, k, \nu \in (-\infty, \infty)$, such that $k/\ell + 1 - \nu \notin \{0, -1, -2, ...\}$. Then, the ℓ-polynomial factorial is defined as*

$$0_\ell^{(\nu)} = 0, \; k_\ell^{(\nu)} = \ell^\nu \frac{\Gamma(k/\ell + 1)}{\Gamma(k/\ell + 1 - \nu)}, \tag{7.3}$$

where Γ is the gamma function and $k_\ell^{(n)} = k(k-\ell)(k-2\ell)\cdots(k-(n-1)\ell)$ if $n \in N$.

Remark 7.2.3. *Note that (i) $e_1(k_{(0)}) = e_1(k)$ and $e_1(k_{(-1)}) = \infty$ if $k > 0$.*

(ii) $e_1(-k_{(1)}) = -\infty$ if $k > 0$ and

(iii) $e_1(k_\ell^{-1}) = 1 + \frac{1}{1!}\frac{1}{(k+\ell)_{-\ell}^{(1)}} + \frac{1}{2!}\frac{1}{(k+\ell)_{-\ell}^{(2)}} + ...\infty$; $\lim_{k\to\infty} e_1(k_\ell^{-1}) = 1$.

Lemma 7.2.4. *If* $k > 0, 0 < \ell < 1$, *then* $e_1(k_{-\ell}) = -e_1((-k)_\ell)$.

Proof. Since $e_1(k_{(-\ell)}) = 1 + \frac{k_{-\ell}^{(1)}}{1!} + \frac{k_{-\ell}^{(2)}}{2!} + \frac{k_{-\ell}^{(3)}}{3!} + \cdots$

$$= 1 + \frac{k}{1!} + \frac{k(k+\ell)}{2!} + \frac{k(k+\ell)(k+2\ell)}{2!} + \cdots$$

$$= -\big[-1 + \frac{-k}{1!} + \frac{-k(-k-\ell)}{2!} + \frac{-k(-k-\ell)(-k-2\ell)}{2!} + \cdots\big]$$

$$= -e_1((-k)_\ell).$$

□

Lemma 7.2.5. *If* $k = m\ell, \ell > 0,\ m \in N(0)$, *then* $e_1(k_\ell)$ *is a finite series such as* $e_1(k_\ell) = \sum\limits_{r=0}^{m} \frac{k_\ell^{(r)}}{r!}$

Proof. Taking $k = m\ell$ in Definition 7.2.1, we have

$$e_1(k_\ell) = 1 + \frac{(m\ell)_\ell^{(1)}}{1!} + \frac{(m\ell)_\ell^{(2)}}{2!} + \frac{(m\ell)_\ell^{(3)}}{3!} + \cdots \tag{7.4}$$

Now $(m\ell)_\ell^{(m+1)} = m\ell(m\ell-\ell)(m\ell-2\ell)\cdots(m\ell-m\ell) = 0$

$$(m\ell)_\ell^{(m+2)} = m\ell(m\ell-\ell)(m\ell-2\ell)\cdots(m\ell-m\ell)(m\ell-(m+1)\ell) = 0.$$

Similarly, we get $(m\ell)_\ell^{(m+s)} = 0$ for $s = 1, 2, ...$

Substituting the above relation in (7.4) gives the proof. □

Lemma 7.2.6. *If* $0 < \ell < 1$, *then* $\Delta_\ell e_1(k_\ell) = \ell e_1(k_\ell)$.

Proof. The proof follows from linearity of Δ_ℓ, (1.1) and $\Delta_\ell k_\ell^{(r)} = r\ell k_\ell^{(r-1)}$. □

Lemma 7.2.7. *If* $k > 0, 0 < \ell < 1$, *then* $\underset{\ell}{\Delta}\, e_1(k_{(\ell)}) = \ell e_1(k_{(\ell)})$.

Proof.
$$\underset{\ell}{\Delta}\, e_1(k_{(\ell)}) = \underset{\ell}{\Delta}\Big[\frac{k_\ell^{(0)}}{0!} + \frac{k_\ell^{(1)}}{1!} + \frac{k_\ell^{(2)}}{2!} + \frac{k_\ell^{(3)}}{3!} + ...\Big]$$
$$= 0 + \frac{\ell(k)_\ell^{(0)}}{1!} + \frac{2\ell(k)_\ell^{(1)}}{2!} + \frac{3\ell(k)_\ell^{(2)}}{3!} + ...$$
$$= \ell\Big[1 + \frac{(k)_\ell^{(1)}}{1!} + \frac{(k)_\ell^{(2)}}{2!} + \frac{(k)_\ell^{(3)}}{3!} + ...\Big] = \ell e_1(k_{(\ell)}).$$
□

Lemma 7.2.8. *If $k > 0, 0 < \ell < 1$, then $\underset{-\ell}{\Delta}\, e_1(k_{(\ell)}) = -\ell e_1((k-\ell)_{(\ell)})$.*

Proof.
$$\underset{-\ell}{\Delta}\, e_1(k_{(\ell)}) = \underset{-\ell}{\Delta}\Big[\frac{k_\ell^{(0)}}{0!} + \frac{k_\ell^{(1)}}{1!} + \frac{k_\ell^{(2)}}{2!} + \frac{k_\ell^{(3)}}{3!} + ...\Big]$$
$$= 0 + \frac{-\ell(k-\ell)_\ell^{(0)}}{1!} + \frac{-2\ell(k-\ell)_\ell^{(1)}}{2!} + \frac{-3\ell(k-\ell)_\ell^{(2)}}{3!} + ...$$
$$= -\ell\Big[1 + \frac{(k-\ell)_\ell^{(1)}}{1!} + \frac{(k-\ell)_\ell^{(2)}}{2!} + \frac{(k-\ell)_\ell^{(3)}}{3!} + ...\Big],$$
which gives the proof. □

Lemma 7.2.9. *If $0 < \ell < 1, n \in N(1)$, then $\Delta_\ell^n e_1(k_\ell) = \ell^n e_1(k_\ell)$.*

Proof. The proof follows by taking Δ_ℓ^n on $e_1(k_\ell)$ and applying the Lemma 7.2.6. □

Definition 7.2.10. *If $k_\ell^{(r)} \neq 0$ for $r = 1, 2, ...,$ then the extorial function is the reciprocal of polynomial factorial and is defined by*

$$e_1(\frac{1}{k_\ell}) = 1 + \frac{1}{1!}\frac{1}{k_\ell^{(1)}} + \frac{1}{2!}\frac{1}{k_\ell^{(2)}} + \frac{1}{3!}\frac{1}{k_\ell^{(3)}} + \cdots \tag{7.5}$$

The extorial function for negative index is defined as

$$e_{-1}(k_\ell) = e_1(k_\ell^{-1}) = \sum_{r=0}^{\infty} \frac{1}{r!}\frac{1}{(k+\ell)_{-\ell}^{(r)}}. \tag{7.6}$$

Note that $e_1\left(\dfrac{1}{k_\ell}\right) \neq e_{-1}(k_\ell)$.

Example 7.2.11. *(i)* $e_1(\frac{1}{1_{-1}}) = \sum_{r=0}^{\infty} \frac{1}{(r!)^2}$ *and (ii)* $e_1(\frac{1}{(-1)_{-1}}) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{(r!)^2}$.

The following Lemma gives the difference formula to extorial function for negative index.

Lemma 7.2.12. *If* $k_\ell^{(r)} \neq 0$ *for* $r = 1, 2, ...,$ *then we have*

$$\Delta_\ell e_1(\frac{1}{k_\ell}) = e_1(\frac{1}{(k-\ell)_\ell})\Delta_\ell (k)_\ell^{-1} = -\frac{\ell}{(k+\ell)_\ell^{(2)}} e_1(\frac{1}{(k-\ell)_{(\ell)}}). \tag{7.7}$$

Proof. From the Definition 7.2.1 and (1.1), we have $\Delta_\ell \frac{1}{k_\ell^{(n)}} = -\frac{n\ell}{(k+\ell)_\ell^{(n+1)}}$ and

$$\begin{aligned}
\Delta_\ell e_1(\tfrac{1}{k_\ell}) &= (1-1) + \frac{1}{1!}\Delta_\ell \frac{1}{k_\ell^{(1)}} + \frac{1}{2!}\Delta_\ell \frac{1}{k_\ell^{(2)}} + + \frac{1}{3!}\Delta_\ell \frac{1}{k_\ell^{(3)}} + \cdots \\
&= \frac{1}{1!}\frac{-\ell}{(k+\ell)_\ell^{(2)}} + \frac{1}{2!}\frac{-2\ell}{(k+\ell)_\ell^{(3)}} + \frac{1}{3!}\frac{-3\ell}{(k+\ell)_\ell^{(3)}} + \cdots \\
&= \frac{-\ell}{(k+\ell)_\ell^{(2)}}\left\{1 + \frac{1}{1!}\frac{1}{(k-\ell)_\ell^{(1)}} + \frac{1}{2!}\frac{1}{(k-\ell)_\ell^{(2)}} + \frac{1}{3!}\frac{1}{(k-\ell)_\ell^{(3)}} + \cdots\right\} \\
&= \frac{-\ell}{(k+\ell)_\ell^{(2)}} e_1(\tfrac{1}{(k-\ell)_\ell}) = e_1(\tfrac{1}{(k-\ell)_\ell})\Delta_\ell (k)_\ell^{-1}.
\end{aligned}$$

□

Corollary 7.2.13. *If* $e_2\left(\frac{1}{k_\ell}\right) = 1 + \frac{1}{1!}\frac{1}{k_\ell^{(2)}} + \frac{1}{2!}\frac{1}{k_\ell^{(4)}} + \frac{1}{3!}\frac{1}{k_\ell^{(6)}} + \cdots,$ *then*

$$\Delta_\ell e_2\left(\frac{1}{k_\ell}\right) = e_2(\frac{1}{(k-2\ell)_\ell})\Delta_\ell (k)_\ell^{-2} = -\frac{-2\ell}{(k+\ell)_\ell^{(3)}} e_2\left(\frac{1}{(k-2\ell)_\ell}\right).$$

Proof. From the Definition 7.2.1 and (1.1), we have

$$\begin{aligned}
\Delta_\ell e_2(\tfrac{1}{k_\ell}) &= (1-1) + \frac{1}{1!}\Delta_\ell \frac{1}{k_\ell^{(2)}} + \frac{1}{2!}\Delta_\ell \frac{1}{k_\ell^{(4)}} + + \frac{1}{3!}\Delta_\ell \frac{1}{k_\ell^{(6)}} + \cdots \\
&= \frac{1}{1!}\frac{-2\ell}{(k+\ell)_\ell^{(3)}} + \frac{1}{2!}\frac{-4\ell}{(k+\ell)_\ell^{(5)}} + \frac{1}{3!}\frac{-6\ell}{(k+\ell)_\ell^{(7)}} + \cdots \\
&= \frac{-2\ell}{(k+\ell)_\ell^{(3)}}\left\{1 + \frac{1}{1!}\frac{1}{(k-2\ell)_\ell^{(2)}} + \frac{1}{2!}\frac{1}{(k-2\ell)_\ell^{(4)}} + \frac{1}{3!}\frac{1}{(k-2\ell)_\ell^{(6)}} + \cdots\right\} \\
&= \frac{-2\ell}{(k+\ell)_\ell^{(3)}} e_{(-2)}((k-2\ell)_{(\ell)}) = e_{(-2)}((k-2\ell)_{(\ell)})\Delta_\ell (k)_\ell^{-2}.
\end{aligned}$$

□

Corollary 7.2.14. *If* $e_m(\frac{1}{k_\ell}) = 1 + \frac{1}{1!}\frac{1}{k_\ell^{(m)}} + \frac{1}{2!}\frac{1}{k_\ell^{(2m)}} + \frac{1}{3!}\frac{1}{k_\ell^{(3m)}} + \cdots$, *then*
$\Delta_\ell e_m\left(\frac{1}{k_\ell}\right) = e_m(\frac{1}{(k-m\ell)_\ell})\Delta_\ell(k)_\ell^{-m} = -\frac{-m\ell}{(k+\ell)_\ell^{(m+1)}}e_{(-m)}((k-m\ell)_{(\ell)}).$

Corollary 7.2.15. *If* $-1 < \ell < 1$ *and* $e_m(k_{(\ell)}) = 1+\frac{1}{1!}k_\ell^{(m)}+\frac{1}{2!}k_\ell^{(2m)}+\frac{1}{3!}k_\ell^{(3m)}+\cdots$, *then* $\Delta_\ell e_m(k_{(\ell)}) = e_m(k_{(\ell)})\Delta_\ell(k)_\ell^m = (m\ell)k_\ell^{(m-1)}e_m(k_{(\ell)})$.

Lemma 7.2.16. *(Product formula) For* $k_1, k_2 \in (-\infty, \infty)$, *we have*

$$e_1((k_1+k_2)_\ell) = e_1((k_1)_\ell)e_1((k_2)_\ell). \tag{7.8}$$

Proof. By replacing k by k_1+k_2 and ν by 1 in (7.2), we arrive

$$\begin{aligned} e_1((k_1+k_2)_\ell) &= 1 + \tfrac{(k_1+k_2)}{1!} + \tfrac{(k_1+k_2)(k_1+k_2-\ell)}{2!} + \tfrac{(k_1+k_2)(k_1+k_2-\ell)(k_1+k_2-2\ell)}{3!} + \cdots + \infty \\ &= 1 + \frac{k_1}{1!} + \frac{k_1(k_1-\ell)}{2!} + \frac{k_1(k_1-\ell)(k_1-2\ell)}{3!} + \cdots + \infty \\ &\quad + \frac{k_2}{1!} + \frac{k_1k_2}{2!} + \frac{k_1k_1^2+k_2k_1^2}{3!} + \cdots + \frac{k_2(k_2-\ell)(k_2-2\ell)}{3!} + \cdots + \infty, \end{aligned}$$

which is same as the RHS of (7.8). □

Example 7.2.17. *By taking* $k_1 = 10, k_2 = 6, \ell = 2$, *we show that*

$e_1((10+6)_2) = e_1(10_2).e_1(6_2)$. *Taking* $k = 16, 10, 6$ *respectively in* (7.2), *we arrive*

$$\begin{aligned} e_1(16_2) = 1+\frac{16}{1!}+\frac{(16)(14)}{2!}+\frac{(16)(14)(12)}{3!}+\frac{(16)(14)(12)(10)}{4!}+\frac{(16)(14)(12)(10)(8)}{5!} \\ +\frac{(16)(14)(12)(10)(8)(6)}{6!}+\frac{(16)(14)(12)(10)(8)(6)(4)}{7!} \\ +\frac{(16)(14)(12)(10)(8)(6)(4)(2)}{8!} = 6561. \end{aligned}$$

$$e_1(10_2) = 1+\frac{10}{1!}+\frac{(10)(8)}{2!}+\frac{(10)(8)(6)}{3!}+\frac{(10)(8)(6)(4)}{4!}+\frac{(10)(8)(6)(4)(2)}{5!} = 243$$

and $e_1(6_2) = 1+\frac{6}{1!}+\frac{(6)(4)}{2!}+\frac{(6)(4)(2)}{3!} = 27 = (243)(27) = 6561.$

which yields $e_1(16_2) = e_1(10_2).e_1(6_2)$.

7.3 Exact solution of heat equation of rod, plate and medium

The extorial function, obtained from exponential function by replacing polynomial into polynomial factorials plays vital role in finding the exact solution of heat equation of rod, thin plate and medium. First time, we have derived this type of exact solution in heat flows. The significance of this function is that when $\ell \to 0$ extorial becomes exponential function which is exact solution of heat flows is continuous case. Through our book it is possible to obtain solutions of difference and differential equations in both discrete and continuous cases.

7.3.1 Extorial function in heat equation of rod

Assume that $v(k_1, k_2)$ be the temperature of a rod at position k_1 at time k_2. Let ℓ_1 and ℓ_2 be the shift values of k_1 and k_2 and γ be the rate of conductivity of rod.

Theorem 7.3.1. *If* $\gamma = \dfrac{\ell_2^n(-\ell_1)^{-n}}{(-1)^n + e_1((-n\ell_1)_{(\ell_1)})}$, *then* $v(k_1, k_2) = e_1(k_{1(\ell_1)})e_1(k_{2(\ell_2)})$ *is a closed form solution of the discrete heat equation model*

$$\underset{(0,\ell_2)}{\Delta^n} v(k_1, k_2) = \gamma \left\{ \underset{(\ell_1,0)}{\Delta^n} v(k_1, k_2) + \underset{(-\ell_1,0)}{\Delta^n} v(k_1, k_2) \right\}, \tag{7.9}$$

where $\underset{(\ell_1,\ell_2)}{\Delta} v(k_1, k_2) = v(k_1 + \ell_1, k_2 + \ell_2) - v(k_1, k_2), n > 0.$

Proof. By Lemma 7.2.7, 7.2.8 we have the relation

$$\underset{(\ell_1,0)}{\Delta} e_1(k_{1(\ell_1)})e_1(k_{2(\ell_2)}) = \ell_1 e_1(k_{1(\ell_1)})e_1(k_{2(\ell_2)}) \tag{7.10}$$

$$\underset{(-\ell_1,0)}{\Delta} e_1(k_{1(\ell_1)})e_1(k_{2(\ell_2)}) = -\ell_1 e_1((k_1-\ell_1)_{(\ell_1)})e_1(k_{2(\ell_2)}). \tag{7.11}$$

In the same way, we get

$$\underset{(\ell_2,0)}{\Delta} e_1(k_{1(\ell_1)})e_1(k_{2(\ell_2)}) = \ell_2 e_1(k_{1(\ell_1)})e_1(k_{2(\ell_2)}). \tag{7.12}$$

Substituting the above equations (7.10), (7.11), (7.12) in (7.9) yields that $\gamma = \dfrac{\ell_2^n(-\ell_1)^{-n}}{(-1)^n + e_1((-n\ell_1)_{(\ell_1)})}$, which completes the proof. $\square$

7.3.2 Extorial function in heat equation of thin plate

Assume that $v(k_1,k_2,k_3)$ be the temperature of a thin plate at position (k_1,k_2) at time k_3. Let ℓ_1,ℓ_2 and ℓ_3 be the shift values of k_1,k_2 and k_3 and γ be the rate of conductivity of thin plate.

Theorem 7.3.2. *If* $\sum\limits_{i=1}^{2}\left[\ell_i^n + (-\ell_i^n)e_1((-n\ell_i)_{(\ell_i)})\right]\gamma = \ell_3^n$, *then* $v(k) = \prod\limits_{i=1}^{3} e_1((k_i)_{(\ell_i)})$ *is an exact solution of the discrete heat equation model*

$$\underset{(0,\ell_3)}{\Delta^n} v(k_1,k_2,k_3) = \gamma\left\{\underset{(\ell_{1,2,0})}{\Delta^n} v(k_1,k_2,k_3) + \underset{(-\ell_{1,2,0})}{\Delta^n} v(k_1,k_2,k_3)\right\}. \tag{7.13}$$

Proof. The proof is similar to the proof of the Theorem (7.3.1). $\square$

7.3.3 Extorial function in heat equation of medium

Assume that $v(k_1, k_2, k_3, k_4, k_5)$ be the temperature of a thin plate at position (k_1, k_2, k_3) at time k_4 and at density k_5. Let (ℓ_1, ℓ_2, ℓ_3) and ℓ_4, ℓ_5 be the shift values of (k_1, k_2, k_3), k_4 and k_5 and γ be the rate of conductivity of medium.

Theorem 7.3.3. *If* $\sum\limits_{i=1}^{3} \Big[\ell_i^n + (-\ell_i^n) e_1((-n\ell_i)_{(\ell_i)})\Big]\gamma = \ell_4^n \ell_5^n$, *then* $v(k) = \prod\limits_{i=1}^{5} e_1((k_i)_{(\ell_i)})$ *is a closed form solution of the discrete heat equation model*

$$\underset{(0,\ell_{4,5})}{\Delta^n} v(k_1, k_2, k_3, k_4, k_5) = \gamma \left\{ \underset{(\pm\ell_{1,2,3})}{\Delta^n} v(k_1, k_2, k_3, k_4, k_5) \right\}. \tag{7.14}$$

Proof. The proof is similar to the proof of the Theorem 7.3.1. □

7.3.4 Extorial function in rod made of multiple materials

Consider a rod made of multiple materials which is non-homogeneous. Then the corresponding heat equation can be expressed as

$$v(k_1, k_2) = \sum_{i \in Z} \alpha(k_1 + r\ell_1) v(k_1 + r\ell_1, k_2 - \ell_2). \tag{7.15}$$

Theorem 7.3.4. *Consider the heat equation (7.15) for a rod made of non-homogeneous multiple materials. If we have the relation*

$$1 = \sum_{i \in Z} \alpha(k_1 + i\ell_1) e_1((-i\ell_2)_{(\pm\ell_2)}).e_1((i\ell_1)_{(\pm\ell_1)}),$$

then we get $v(k_1, k_2) = e_1((k_1)_{(\pm\ell_1)}) e_1((k_2)_{(\pm\ell_2)})$ *is a closed form solution of* (7.15).

Proof. Taking $v(k_1,k_2)=e_1((k_1)_{(\pm\ell_1)})e_1((k_2)_{(\pm\ell_2)})$ in (7.15) for three materials,

$$e_1((k_1)_{(\pm\ell_1)})e_1((k_2)_{(\pm\ell_2)})=e_1((k_1)_{(\pm\ell_1)})e_1((k_2)_{(\pm\ell_2)})\Big[\alpha(k_1)e_1((-\ell_2)_{(\pm\ell_2)})+\alpha(k_1+\ell_1)$$
$$e_1((\ell_1)_{(\pm\ell_1)}).e_1((-\ell_2)_{(\pm\ell_2)})+\alpha(k_1-\ell_1)e_1((-\ell_1)_{(\pm\ell_1)}).e_1((-\ell_2)_{(\pm\ell_2)})\Big].$$

Cancelling $e_1((k_1)_{(\pm\ell_1)}).e_1((k_2)_{(\pm\ell_2)})$ on both sides, we get the result for three materials. By induction on number of multiple materials, we get (7.15). □

Special cases:

(i) If $\ell_1=\ell_2$, then we have $1=\sum\limits_{i\in Z}\alpha(k_1+i\ell_1)e_1((-i\ell_2)_{(\pm\ell_1)}).e_1((i\ell_1)_{(\pm\ell_1)})$.

(ii) If $\alpha(k_1+i\ell_1)=\dfrac{1}{3}$, then we have $3=\sum\limits_{i\in Z}\Big[e_1((-i\ell_2)_{(\pm\ell_2)}).e_1((i\ell_1)_{(\pm\ell_1)})\Big]$.

7.4 Extorial solution of linear difference equation

Consider a second order linear difference equation with constant coefficients $\Big[a\Delta_\ell^2+b\Delta_\ell+c\Big]u(k)=e^{sk}$, where s is a constant and k is a variable with respect to shift value ℓ. Now, consider the second order homogeneous equation

$$\Big[a\Delta_\ell^2+b\Delta_\ell+c\Big]u(k)=0. \tag{7.16}$$

Dividing the above equation by ℓ^2 on both sides and let us take $A=a$, $B=\dfrac{b}{\ell}$ and $C=\dfrac{c}{\ell^2}$ we have

$$\Big[A\frac{\Delta_\ell^2}{\ell^2}+B\frac{\Delta_\ell}{\ell}+C\Big]u(k)=0. \tag{7.17}$$

If we assume that $u(k) = e_1((mk)_{(m\ell)})$ be a solution of (7.17).

$A\frac{\Delta_\ell^2}{\ell^2}e_1((mk)_{(m\ell)}) + B\frac{\Delta_\ell}{\ell}e_1((mk)_{(m\ell)}) + Ce_1((mk)_{(m\ell)}) = 0.$

$$\frac{A}{\ell^2}\Delta_\ell^2 e_1((mk)_{(m\ell)}) + \frac{B}{\ell}\Delta_\ell e_1((mk)_{(m\ell)}) + Ce_1((mk)_{(m\ell)}) = 0. \tag{7.18}$$

Now, $\Delta_\ell e_1((mk)_{(m\ell)}) = 0 + \frac{(mk)^{(0)}_{(m\ell)}}{1!} + \frac{2m\ell(mk)^{(1)}_{(m\ell)}}{2!} + \frac{3m\ell(mk)^{(2)}_{(m\ell)}}{3!} + ...$

$$= m\ell\Big[1 + \frac{(mk)^{(1)}_{(m\ell)}}{1!} + \frac{(mk)^{(2)}_{(m\ell)}}{2!} + ...\Big] = (m\ell)e_1((mk)_{(m\ell)}).$$

$$\Delta_\ell^2 e_1((mk)_{(m\ell)}) = (m\ell)\Delta_\ell e_1((mk)_{(m\ell)}) = (m\ell)^2 e_1((mk)_{(m\ell)}).$$

From (7.18), $\frac{A}{\ell^2}(m\ell)^2 e_1((mk)_{(m\ell)}) + \frac{B}{\ell}(m\ell)e_1((mk)_{(m\ell)}) + Ce_1((mk)_{(m\ell)}) = 0$

$$\Big[\frac{A}{\ell^2}(m\ell)^2 + \frac{B}{\ell}(m\ell) + C\Big]e_1((mk)_{(m\ell)}) = 0$$

which yields $Am^2 + Bm + C = 0$.

If particularly, $u(k) = e^{sk}$ then equation (7.16) results

$$[a\Delta_\ell^2 + b\Delta_\ell + C]\Big\{\frac{e^{sk}}{a(e^{sl}-1)^2 + b(e^{sl}-1) + c}\Big\} = e^{sk}. \tag{7.19}$$

From (7.16) and (7.19), then $u(k) = \dfrac{e^{sk}}{a(e^{sl}-1)^2 + b(e^{sl}-1) + c}$.

Case (i): Let m_1 and m_2 be the two distinct roots of (7.17) then the complementary function is $u(k) = C_1e_1((m_1k)_{(m_1\ell)}) + C_2e_1((m_2k)_{(m_2\ell)})$.

The solution of the equation is

$$u(k) = C_1e_1((m_1k)_{(m_1\ell)}) + C_2e_1((m_2k)_{(m_2\ell)}) + \frac{e^{sk}}{a(e^{sl}-1)^2 + b(e^{sl}-1) + c}. \tag{7.20}$$

Example 7.4.1. *Let us take the difference equation* $(\Delta_\ell^2 - 5\Delta_\ell + 6)u(k) = e^k$, $\ell = 1$, $s = 1$ *then the roots of the equation are* $m_1 = 3$, $m_2 = 2$ *which are distinct and* $u(k) = \dfrac{e^k}{(e^1-1)^2 - 5(e^1-1) + 6}$. *From the given equation* $a = 1$, $b = -5$, $c = 6$.

The solution is $u(k) = C_1 e_1((3k)_{(3\ell)}) + C_2 e_1((2k)_{(2\ell)}) + \dfrac{e^k}{0.3610833}$

$$u(0) = C_1 + C_2 + 2.769444$$

$$\Delta_\ell u(k) = C_1 \Delta_\ell e_1((3k)_{(3\ell)}) + C_2 \Delta_\ell e_1((2k)_{(2\ell)}) + \Delta_\ell \frac{e^k}{0.3610833}$$

$$\Delta_\ell u(0) = 3C_1 + 2C_2 + 2.769444.$$

From $u(0)$ *and* $\Delta_\ell u(0)$, *we get* $C_1 = 0.308332$, $C_2 = -0.077776$.

Case (ii): Let m_1 and m_2 be the same distinct roots of (7.17) then complementary function is $u(k) = C_1 e_1((mk)_{(m\ell)}) + C_2 e_1((mk)_{(m\ell)}) e_1((mk)_{(m\ell)})$. The solution is

$$u(k) = \left[C_1 + C_2 e_1((mk)_{(m\ell)})\right] e_1((mk)_{(m\ell)}) + \frac{e^{sk}}{a(e^{sl}-1)^2 + b(e^{sl}-1) + c}. \tag{7.21}$$

Example 7.4.2. *Let us take the difference equation* $(\Delta_\ell^2 - 6\Delta_\ell + 9)u(k) = e^k$, $\ell = 1$, $s = 1$ *then the roots of the equation are* $m_1 = 3$, $m_2 = 3$ *which are same and* $u(k) = \dfrac{e^k}{(e^1-1)^2 - 5(e^1-1) + 6}$. *From the given equation* $a = 1$, $b = -6$, $c = 9$.

The solution is $u(k) = \left[C_1 + C_2 e_1((3k)_{(3\ell)})\right] e_1((3k)_{(3\ell)}) + \dfrac{e^k}{0.3610833}$

$$u(0) = C_1 + C_2 + 2.769444$$

$$\Delta_\ell u(k) = \left[C_1 + C_2 \Delta_\ell e_1((3k)_{(3\ell)})\right] \Delta_\ell e_1((3k)_{(3\ell)}) + \Delta_\ell \frac{e^k}{0.3610833}$$

$$\Delta_\ell u(0) = 3C_1 + 9C_2 + 2.769444$$

From $u(0)$ *and* $\Delta_\ell u(0)$, *we get* $C_1 = -0.859259$, $C_2 = 0.089815$.

7.5 Exact solutions of RL circuits

The resistor and inductor are the most fundamental linear (element having linear relationship between voltage and current) and passive elements in electric circuits. When resistor and inductor are connected across voltage supply, the circuit so obtained is called as RL circuit which can be either in a series or parallel circuit depending on the nature of connection between the resistor and inductor. The extorial function act as exact solution of RL circuits. Here, we obtain exact solutions of RL circuits.

7.5.1 Current flows in RL circuit

Consider RL circuit, by using the Kirchhoff's circuit rule, the differential equation connecting voltage V, resistance R, current I and induction L in series is given by

$$V = RI(k) + L\frac{dI(k)}{dk}. \tag{7.22}$$

Now replacing $dI(k) = \Delta_\ell I(k)$, where $\Delta_\ell I(k) = I(k+\ell) - I(k)$ and $dk = \ell$ in (7.22), the corresponding difference equation for the current flows in RL series circuit in the discrete case is

$$V = RI(k) + L\frac{\Delta_\ell I(k)}{\ell}. \tag{7.23}$$

While heat appears in the RL circuit, the difference equation (7.23) becomes fractional difference equation

$$V = RI(k) + L\frac{\Delta_\ell^\nu I(k)}{\ell^\nu}, (0 < \nu < 1). \tag{7.24}$$

The corresponding fractional difference equation for de-energizing in RL circuit is

$$0 = RI(k) + L\frac{\Delta_\ell^\nu I(k)}{\ell^\nu}. \tag{7.25}$$

Finding the solution of the discrete and fractional difference equations (7.23) and (7.24) is another aim of this book. We obtain solution for the equations (7.23) and (7.24) using extorial functions.

7.5.2 Extorial Solution of RL circuit

In this section, we find solution of equation (7.23) after arriving at some basic results of extorial functions. This extorial function is easily obtained by replacing polynomial k^n by $k_\ell^{(n)}$ in the expansion of exponential function e^k. This function is useful to arrive solutions for fractional difference equation.

Definition 7.5.1. *The extorial function $e_1((mk)_{(m\ell)})$ is defined by*

$$e_1((mk)_{(m\ell)}) = 1 + \frac{(mk)_{m\ell}^{(1)}}{1!} + \frac{(mk)_{m\ell}^{(2)}}{2!} + \cdots + \infty = \sum_{r=0}^{\infty} \frac{(mk)_{m\ell}^{(r)}}{r!}, \tag{7.26}$$

where $(mk)_{m\ell}^{(r)} = (mk)(mk - m\ell)\cdots(mk - (r-1)m\ell)$ for positive integer r, is the *falling polynomial factorial. In general, for real index* ν,

$$e_{(\nu)}((mk)_{(m\ell)}) = 1 + \frac{(mk)^{(\nu)}_{m\ell}}{1!} + \frac{(mk)^{(2\nu)}_{m\ell}}{2!} + \cdots + \infty = \sum_{r=0}^{\infty} \frac{(mk)^{(r\nu)}_{m\ell}}{r!}, \tag{7.27}$$

where $(mk)^{(r\nu)}_{m\ell} = (m\ell)^{(r\nu)} \dfrac{\Gamma(\frac{k}{\ell}+1)}{\Gamma(\frac{k}{\ell}+1-r\nu)}$ *and* $\Gamma(.)$ *is the gamma function.*

Lemma 7.5.2. *If* $e_1((mk)_{(m\ell)})$ *is the extorial function, then we have*

$\Delta_\ell e_1((mk)_{(m\ell)}) = (m\ell)e_1((mk)_{(m\ell)})$.

Proof. Applying Δ_ℓ on the extorial function $e_1((mk)_{(m\ell)})$, we arrive

$$\begin{aligned}
\Delta_\ell e_1((mk)_{(m\ell)}) &= \Delta_\ell(1) + \Delta_\ell \frac{(mk)^{(1)}_{m\ell}}{1!} + \Delta_\ell \frac{(mk)^{(2)}_{m\ell}}{2!} + \cdots + \infty \\
&= 0 + \frac{(m(k+\ell))^{(1)}_{m\ell}}{1!} + \frac{(mk)^{(1)}_{m\ell}}{1!} + \frac{1}{2!}\Big[(m(k+\ell))^{(2)}_{m\ell} - (mk)^{(2)}_{m\ell}\Big] + \cdots + \infty \\
&= \frac{(m\ell)}{1!} + \frac{1}{2!}\Big[(mk+m\ell)(mk) - (mk)(mk-m\ell)\Big] + \cdots + \infty \\
&= m\ell + \frac{2m\ell}{2!}(mk)^{(1)}_{m\ell} + \frac{3m\ell}{3!}(mk)^{(2)}_{m\ell} + \cdots + \infty \\
&= m\ell\Big[1 + \frac{(mk)^{(k)}_{m\ell}}{1!} + \frac{(mk)^{(2)}_{m\ell}}{2!} + \cdots + \infty\Big]
\end{aligned}$$

$\Delta_\ell e^{(mk)_{m\ell}} = (m\ell)e_1((mk)_{(m\ell)})$. □

Lemma 7.5.3. *The extorial function* $u(k) = e_1((mk)_{(m\ell)})$ *is a solution of equation*

$$\Big(A\frac{\Delta_\ell^2}{\ell^2} + B\frac{\Delta_\ell}{\ell} + C\Big)u(k) = 0, \tag{7.28}$$

if m *is a root of the auxiliary equation* $Am^2 + Bm + C = 0$.

Proof. If we try $u(k) = e_1((mk)_{(m\ell)})$ as a solution of equation(7.28), then it should satisfy the equation

$$\frac{A}{\ell^2}\Delta_\ell^2 e_1((mk)_{(m\ell)}) + \frac{B}{\ell}\Delta_\ell e_1((mk)_{(m\ell)}) + Ce_1((mk)_{(m\ell)}) = 0. \tag{7.29}$$

Using (iv) of Lemma 7.5.2, linear property of Δ_ℓ and the expansion of $e_1((mk)_{(m\ell)})$, we arrive at

$$\Delta_\ell e_1((mk)_{(m\ell)}) = 0 + (m\ell)\frac{(mk)^{(0)}_{(m\ell)}}{1!} + \frac{2m\ell(mk)^{(1)}_{(m\ell)}}{2!} + \frac{3m\ell(mk)^{(2)}_{(m\ell)}}{3!} + ...$$

$$\text{i.e, } \Delta_\ell e_1((mk)_{(m\ell)}) = m\ell\Big[1 + \frac{(mk)^{(1)}_{(m\ell)}}{1!} + \frac{(mk)^{(2)}_{(m\ell)}}{2!} + ...\Big] = m\ell e_1((mk)_{(m\ell)}),$$

which yields $\Delta^2_\ell e_1((mk)_{(m\ell)}) = (m\ell)\Delta_\ell e_1((mk)_{(m\ell)}) = (m\ell)^2 e_1((mk)_{(m\ell)})$.

Applying the values of $\Delta_\ell e_1((mk)_{(m\ell)})$ and $\Delta^2_\ell e_1((mk)_{(m\ell)})$ in (7.29), we obtain

$$\Big(\frac{A}{\ell^2}(m\ell)^2 + \frac{B}{\ell}(m\ell) + C\Big)e_1((mk)_{(m\ell)}) = 0$$

$$\text{which yields, } Am^2 + Bm + C = 0. \tag{7.30}$$

Hence $u(k) = e_1((mk)_{(m\ell)})$ is a solution of (7.28) when m is a root of (7.29). □

Remark 7.5.4. *The above lemma can be extended to higher order linear difference equation with constant coefficients.*

Theorem 7.5.5. *Let I_0 be initial value of $I(t)$. The de-energizing difference equation for $\nu = 1$, has a solution of the form*

$$I(k) = I_0 e_1\left(\left(\frac{-R}{L}k\right)_{(\frac{-R}{L}\ell)}\right). \tag{7.31}$$

Proof. The auxiliary equation $mL+R = 0$ of equation (7.25) has an unique solution $m = \dfrac{-R}{L}$, when $L \neq 0$. Applying Lemma 7.5.3 for first order difference yields, $I(t) = I_0 e_1((\frac{-R}{L}k)_{(\frac{-R}{L}\ell)})$, as a solution of (7.25) for $\nu = 1$. □

Theorem 7.5.6. *For $\nu = 1$, the energizing difference equation (7.24) has a solution*

$$I(k) = \frac{V}{\frac{L}{\ell}(e^{s\ell} - 1) + R} + I_0 e_1 \left(\left(\frac{-R}{L} k \right)_{(\frac{-R}{L}\ell)} \right). \tag{7.32}$$

Proof. Let $I(k) = \frac{V}{C} e^{sk}$ be a solution of equation (7.24), $\nu = 1$, where C is to be determined. Since s is a constant, from (7.32), we get $\Delta_\ell e^{sk} = e^{s(k+\ell)} - e^{sk}$ which gives a difference equation of the form $\Delta_\ell I(k) = \frac{V}{C} \Delta_\ell e^{sk} = \frac{V}{C} e^{sk}(e^{sl} - 1)$. Substituting $\nu = 1$, $I(k)$ and $\Delta_\ell I(k)$ in (7.24), we arrive

$$I(k)R + L\frac{\Delta_\ell I(k)}{\ell} = R\frac{V}{C}e^{sk} + \frac{L}{\ell}\Big[\frac{V}{C}e^{sk}(e^{sl} - 1)\Big],$$

which yields $\Big[R + \frac{L}{\ell}\Delta\Big] I = \frac{V}{C}\Big[\frac{L}{\ell}(e^{s\ell} - 1 + R)\Big] e^{sk}$.
Hence, taking $C = \frac{L}{\ell}(e^{s\ell} - 1 + R)$, we find $I(k) = \frac{V}{\frac{L}{\ell}(e^{s\ell} - 1 + R)} e^{sk}$ is a particular solution of equation (7.24) when $\nu = 1$.

Now (7.32) follows by adding (7.31) and the above particular solution. □

Corollary 7.5.7. *If $I_0 = \frac{-V}{R}$, then the extorial solution of difference equation (7.23) of the RL circuit is $I(k) = \frac{V}{R} - \frac{V}{R} e_1 \left(\left(\frac{-R}{L} k \right)_{(\frac{-R}{L}\ell)} \right)$.*

Proof. The proof follows by taking $s = 0$ in (7.32). □

7.5.3 Extorial energizing for RL circuit

In this section, we derive solution of RL circuit model with extorial energizing.

Theorem 7.5.8. *The flow of current in the RL circuit creates chaos due to generation of heat. In this case, the difference equation of RL circuit is*

$$Ve_1((sk)_{s\ell}) = RI(k) + L\frac{\Delta_\ell^\nu I(k)}{\ell^\nu}, (0 < \nu < 1). \tag{7.33}$$

Equation (7.33) is ν^{th} order fractional difference equation. When there is no choas in RL circuit, the parameter ν takes integer value.

Theorem 7.5.9. *For $\nu = 1$, energizing difference equation $Ve_1((sk)_{s\ell}) = RI(k) + L\frac{\Delta_\ell I(k)}{\ell}$ has a solution*

$$I(k) = \frac{Ve_1((sk)_{s\ell})}{\frac{L}{\ell}(e^{s\ell} - 1) + R} + I_0 e_1\Big(\Big(\frac{-R}{L}k\Big)_{(\frac{-R}{L}\ell}\Big). \tag{7.34}$$

Proof. Let $I(k) = \frac{V}{C}e_1((sk)_{s\ell})$ be a solution of equation ($\nu = 1$), where C is to be determined. Since s is a constant, from (7.34), we get

$\Delta_\ell e_1((sk)_{s\ell}) = e_1((s(k+\ell))_{(s\ell)}) - e_1((sk)_{s\ell})$ which gives

$$\Delta_\ell I(k) = \frac{V}{C}\Delta_\ell e_1((sk)_{s\ell}) = \frac{V}{C}e_1((sk)_{(s\ell)})(e_1((\ell)_{(s\ell)}) - 1).$$

Substituting $\nu = 1$, $I(k)$ and $\Delta_\ell I(k)$ in the above equation, we arrive

$$I(k)R + L\frac{\Delta_\ell I(k)}{\ell} = R\frac{V}{C}e_1((sk)_{s\ell}) + \frac{L}{\ell}\Big[\frac{V}{C}e^{(sk)}(e_1(\ell_{(s\ell)}) - 1)\Big],$$

which yields $\Big[R + \frac{L}{\ell}\Delta\Big]I = \frac{V}{C}\Big[\frac{L}{\ell}(e_1(\ell_{(s\ell)}) - 1 + R)\Big]e_1((sk)_{s\ell})$.

Hence taking $C = \frac{L}{\ell}(e_1(\ell_{(s\ell)}) - 1 + R)$, we find $I(k) = \dfrac{V}{\frac{L}{\ell}(e_1(\ell_{(s\ell)}) - 1 + R)}e^{sk}$ is a particular solution of equation when $\nu = 1$ and (7.34) follows. □

Theorem 7.5.10. *For $0 < \nu < 1$, the energizing fractional difference equation*

$$Ve_1((sk)_{(s\ell)}) = I(k)R + \frac{\Delta_\ell^\nu I(k)}{\ell}, \tag{7.35}$$

has an extorial solution of the form

$$\frac{Ve_1((sk)_{(s\ell)})}{\frac{L}{\ell}(e_1(\ell_{(s\ell)}) - 1)^\nu + R} + I_0 e_1((\frac{-R}{L})^{\frac{1}{\nu}} k_{(\frac{-R}{L})^{\frac{1}{\nu}}\ell}). \tag{7.36}$$

Proof. We try $I(k) = VCe_1((sk)_{(s\ell)})$ as a solution of equation (7.35), where C is to be determined.

$\Delta_\ell I(k) = VC(e_1(\ell_{(s\ell)}) - 1)e_1((sk)_{(s\ell)})$, $\Delta_\ell^2 I(k) = VC(e_1(\ell_{(s\ell)}) - 1)^2 e_1((sk)_{(s\ell)}) \cdots$,

$\Delta_\ell^\nu I(k) = VC(e_1(\ell_{(s\ell)}) - 1)^\nu e_1((sk)_{(s\ell)})$ is obtained from $\Delta_\ell I(k) = I(k+\ell) - I(k)$.

Substituting I and $\Delta_\ell^\nu I$ in (7.35), we find

$$\begin{aligned} I(k)R + \frac{L}{\ell}\Delta_\ell^\nu I(k) &= RVCe_1((sk)_{(s\ell)}) + \frac{L}{\ell}\Big[VC(e_1(\ell_{(s\ell)}) - 1)^\nu e_1((sk)_{(s\ell)})\Big] \\ &= VC\Big[\frac{L}{\ell}(e_1(\ell_{(s\ell)}) - 1)^\nu + R\Big]e_1((sk)_{(s\ell)}). \end{aligned}$$

Hence $I(k) = \dfrac{V}{\frac{L}{\ell}(e_1(\ell_{(s\ell)}) - 1)^\nu + R} e_1((sk)_{(s\ell)})$ is a particular solution of (7.36). $\square$

7.6 Fractional difference heat equation model

In this section, we apply the alpha and Fibonacci difference operators and obtain new model of heat equations. The solution of these equations are expressed in terms of extorial functions. The materials up to three dimensions i.e., rod, thin plate and medium are taken for study and the transfer of heat is examined. The two operators (alpha and Fibonacci) are used for the study of transfer of heat and are defined accordingly.

7.6.1 Extorial solutions of alpha-Fibonacci difference equation

Let $\alpha \neq 0,\ l = (\ell_1, \ell_2, \ell_3, ..., \ell_n) \neq 0,\ k = (k_1, k_2, \cdots, k_n) \in \mathbb{R}^n$ and $v(k)$ be a real valued n-variable function defined on $\mathbb{R}^n$. The n-variable α-difference operator, denoted as $\underset{\alpha(\ell)}{\Delta}$, on $v(k)$ is defined by

$$\underset{\alpha(\ell)}{\Delta} v(k) = v(k_1 + \ell_1, k_2 + \ell_2, ..., k_n + \ell_n) - \alpha v(k_1, k_2, ..., k_n). \tag{7.37}$$

This operator becomes partial α-difference operator if some $\ell_i = 0$ but not all ℓ_j. Thus the above definition of the alpha and Fibonacci difference operators and its equations are employed in the forthcoming sections and solutions are derived for heat equations.

7.6.2 Alpha difference equation and its solution

In this section, we present solutions of partial alpha difference equation with polynomial factorial and extorial functions. We also apply these type of solutions to heat flows. In the following lemma some identities related to alpha difference operator on extorial function are given.

Lemma 7.6.1. *Let* $k_\ell^{(rn)} \neq 0$, $n \in N$, $-1 < \ell < 1, \ell \neq 0$. *Then we have the following identities with extorial function:*

(i). $\underset{\alpha(\ell)}{\Delta} e_1(k_\ell) = e_1(k_\ell)[1 + \ell - \alpha]$,

(ii). $\underset{\alpha(\ell)}{\Delta} e_{(-1)}(k_\ell) = e_{(-1)}(k_\ell)[e_{(-1)}(\ell_\ell) - \alpha]^{(-1)}$,

(iii). $\underset{\alpha(\ell)}{\Delta} e_1((-k)_\ell) = e_1((-k)_\ell)[1 + \ell - \alpha]$, $k > 0$.

Proof. (i). By (7.37), and applying $\underset{\alpha(\ell)}{\Delta}$ on $e_1(k_\ell)$, we arrive

$$\underset{\alpha(\ell)}{\Delta} e_1(k_\ell) = e_1((k+\ell)_\ell) - \alpha e_1(k_\ell) = e_1(k_\ell).e_1(\ell_\ell) - \alpha e_1(k_\ell) = e_1(k_\ell)[e_1(\ell_\ell) - \alpha]$$

$$= e_1(k_\ell)[1 + \frac{\ell_\ell^{(1)}}{1!} + \frac{\ell_\ell^{(2)}}{2!} + \cdots - \alpha] = e_1(k_\ell)[1 + \ell - \alpha].$$

(ii). By (7.37), and applying $\underset{\alpha(\ell)}{\Delta}$ on $e(k_\ell^{(-1)})$, we arrive

$$\underset{\alpha(\ell)}{\Delta} e_{(-1)}(k_\ell) = e_{(-1)}((k+\ell)_\ell) - \alpha e_{(-1)}(k_\ell) = e_{(-1)}(k_\ell).e_{(-1)}(\ell_\ell) - \alpha e_{(-1)}(k_\ell).$$

$$= e_{(-1)}(k_\ell)[e_{(-1)}(\ell_\ell) - \alpha]^{(-1)}.$$

(iii). follows from (ii) by replacing k as $-k$. □

Theorem 7.6.2. *If* $v(k_1, k_2) = e_1((k_1)_{\ell_1}).e_1((k_2)_{\ell_2})$ *then we have the identities:*

(i) $\underset{\alpha(0,\ell_2)}{\Delta} v(k_1, k_2) = e_1((k_1)_{\ell_1}).e_1((k_2)_{\ell_2})\Big[e_1((\ell_2)_{\ell_2}) - \alpha\Big]$,

(ii) $\underset{\alpha(\ell_1,0)}{\Delta} v(k_1, k_2) = e_1((k_2)_{\ell_2}).e_1((k_1)_{\ell_1})\Big[e_1((\ell_1)_{\ell_1}) - \alpha\Big]$.

Proof. *(i)* $\underset{\alpha(0,\ell_2)}{\Delta} v(k_1, k_2) = e_1((k_1)_{\ell_1})\Big[\underset{\alpha(0,\ell_2)}{\Delta} e_1((k_2)_{\ell_2})\Big]$

$$= e_1((k_1)_{\ell_1})\Big[e_1((k_2+\ell_2)_{\ell_2}) - \alpha e_1((k_2)_{\ell_2})\Big]$$

$$= e_1((k_1)_{\ell_1})e_1((k_2)_{\ell_2})\Big[e_1((\ell_2)_{\ell_2}) - \alpha\Big].$$

In the similar way, the proof of (ii) follows. □

Assume that $v(k_1, k_2)$ be the temperature of a rod at position k_1 at time k_2, ℓ_1 and ℓ_2 be shift values of k_1 and k_2 respectively and γ be the rate of conductivity of rod. When considering impact of external climate change on the rod, the partial α - difference equation of heat flow in the rod becomes fractional α- difference equation

$$\underset{\alpha(0,\ell_2)}{\Delta^\nu} v(k_1, k_2) = \gamma\Big[\underset{\alpha(\ell_1,0)}{\Delta^\nu} v(k_1, k_2) + \underset{\alpha(-\ell_1,0)}{\Delta^\nu} v(k_1, k_2)\Big]. \tag{7.38}$$

Theorem 7.6.3. *If* $\gamma = \Big[e_1((\ell_2)_{\ell_2}) - \alpha/e_1(\pm(\ell_1)_{\ell_1}) - \alpha\Big]$, *then the function* $v(k_1, k_2) = e_1((k_1)_{\ell_1}).e_1((k_2)_{\ell_2})$ *is exact solution of the* α*- difference equation* (7.38).

Proof. By applying the Theorem 7.6.2 and (7.8), we get the proof. □

Corollary 7.6.4. *The fractional partial* α*-difference heat equation* (7.38) *has a solution* $v(k_1, k_2) = e_1((k_1)_{\ell_1}).e_1((k_2)_{\ell_2})$ *if* $\gamma = \Big[(e_1((\ell_2)_{\ell_2}) - \alpha)^\nu/(e_1(\pm(\ell_1)_{\ell_1}) - \alpha)^\nu\Big]$.

Assume that $v(k_1, k_2, k_3)$ be the temperature of a thin plate at position (k_1, k_2) at time k_3. Let (ℓ_1, ℓ_2) and ℓ_3 be the shift values of (k_1, k_2) and k_3 and γ be the rate of conductivity of thin plate. The fractional partial α-difference heat equation of thin plate is given by

$$\underset{\alpha(\ell_3)}{\Delta^\nu} v(k_1, k_2, k_3) = \gamma\left\{\underset{\alpha(\ell_{1,2})}{\Delta^\nu} v(k_1, k_2, k_3) + \underset{\alpha(-\ell_{1,2})}{\Delta^\nu} v(k_1, k_2, k_3)\right\}. \tag{7.39}$$

Corollary 7.6.5. *If* $\gamma = \Big[1 + \ell_3 - \alpha)^\nu/(e_1(\pm(\ell_{1,2})_{(\ell_{1,2})}) - \alpha)^\nu\Big]$, *then the function* $v(k) = \prod_{i=1}^{3} e_1(k_{i(\ell_i)})$ *is an exact solution of the fractional partial heat equation* (7.39).

Assume that $v(k_1, k_2, k_3, k_4, k_5)$ be the temperature of a medium at position (k_1, k_2, k_3) at time k_4 and at density k_5. Let (ℓ_1, ℓ_2, ℓ_3) and ℓ_4, ℓ_5 be the shift values of (k_1, k_2, k_3), k_4 and k_5 and γ be the rate of conductivity of medium. The fractional partial α-difference equation of heat flow in medium is

$$\underset{\alpha(\ell_{(4,5)})}{\Delta^{\nu}} v(k_1, k_2, k_3, k_4, k_5) = \gamma \left\{ \underset{\alpha(\pm \ell_{(1,2,3)})}{\Delta^{\nu}} v(k_1, k_2, k_3, k_4, k_5) \right\}. \tag{7.40}$$

Corollary 7.6.6. *If* $\gamma = \Big[1 + \ell_4 - \alpha)^{\nu} + e_1(1 + \ell_5 - \alpha)^{\nu} / (e_1(\pm(\ell_{1,2,3})_{(\ell_{1,2,3})}) - \alpha)^{\nu}\Big]$, *then* $v(k) = \prod_{i=1}^{5} e_1(k_{i(\ell_i)})$ *is a closed form solution of the fractional partial* α*-difference equation* (7.40).

7.6.3 Fibonacci difference equation and its solution

In this section, we present solution of partial difference equation with polynomial factorial and extorial functions. We also apply these type of solutions to heat flows. Fibonacci difference equation is an extension of alpha difference equation. The x-Fibonacci difference operator is defined in (5.1).

Lemma 7.6.7. *Let* $k_{\ell}^{(rn)} \neq 0$, $n \in N$, $-1 < \ell < 1, \ell \neq 0$. *Then we have*

(i). $\underset{x(\ell)}{\Delta} e_1(k_\ell) = e_1(k_\ell)[1 - x_1 e_1(-\ell_\ell) - x_2 e_1(-2\ell_\ell)]$,

(ii). $\underset{x(\ell)}{\Delta^{\nu}} e_1(k_\ell) = e_1(k_\ell)[1 - x_1 e_1(-\ell_\ell) - x_2 e_1(-2\ell_\ell)]^{\nu}$.

Proof. (i). By (5.1), and applying $\underset{x(\ell)}{\Delta}$ on $e^{k_{\ell}^{(1)}}$, we arrive

$\underset{x(\ell)}{\Delta} e_1(k_\ell) = e_1(k_\ell) - x_1 e_1((k - \ell)_\ell) - x_2 e_1((k - 2\ell)_\ell)$

$$= e_1(k_\ell) - x_1 e_1(k_\ell) e_1(-\ell_\ell) - x_2 e_1(k_\ell) e_1(-2\ell_\ell)$$

$$= e_1(k_\ell)[1 - x_1 e_1(-\ell_\ell) - x_2 e_1(-2\ell_\ell)].$$

Similarly, we get the proof of (ii). □

Theorem 7.6.8. *If $v(k_1, k_2) = e_1((k_1)_{\ell_1}).e_1((k_2)_{\ell_2})$, then we have the identities*

(i). $\underset{x(0,\ell_2)}{\Delta} v(k_1, k_2) = e_1((k_1)_{\ell_1}).e_1((k_2)_{\ell_2})\Big[1 - x_1 e_1((\ell_2)_{\ell_2}) - x_2 e_1((2\ell_2)_{\ell_2})\Big]$,

(ii) $\underset{x(\ell_1,0)}{\Delta} v(k_1, k_2) = e_1((k_2)_{\ell_2}).e_1((k_1)_{\ell_1})\Big[1 - x_1 e_1((\ell_1)_{\ell_1}) - x_2 e_1((2\ell_1)_{\ell_1})\Big]$.

Proof. *(i).*
$$\underset{x(0,\ell_2)}{\Delta} v(k_1, k_2) = e_1((k_1)_{\ell_1})\Big[\underset{x(0,\ell_2)}{\Delta} e_1((k_2)_{\ell_2})\Big]$$
$$= e_1((k_1)_{\ell_1})\Big[1 - x_1 e_1((k_2 + \ell_2)_{\ell_2}) - x_2 e_1((k_2 + 2\ell_2)_{\ell_2})\Big]$$
$$= e_1((k_1)_{\ell_1}) e_1((k_2)_{\ell_2})\Big[1 - x_1 e_1((\ell_2)_{\ell_2}) - x_2 e_1((2\ell_2)_{\ell_2})\Big].$$

In the similar way, the proof of (ii) follows. □

Assume that $v(k_1, k_2)$ be the temperature of a rod at position k_1 at time k_2, ℓ_1 and ℓ_2 be shift values of k_1 and k_2 respectively and γ be the rate of conductivity of rod. When considering impact of external climate change on the rod, the partial x - difference equation of heat flow in the rod becomes fractional x-Fibonacci difference equation

$$\underset{x(0,\ell_2)}{\Delta^\nu} v(k_1, k_2) = \gamma\Big[\underset{x(\ell_1,0)}{\Delta^\nu} v(k_1, k_2) + \underset{x(-\ell_1,0)}{\Delta^\nu} v(k_1, k_2)\Big], \tag{7.41}$$

where $\underset{x(\ell_1,\ell_2)}{\Delta} v(k_1, k_2) = v(k_1 + \ell_1, k_2 + \ell_2) - x_1 v(k_1, k_2) - x_2 v(k_1, k_2)$, $0 < \nu < 1$. Hence, we need to find out solution of fractional difference equation (7.41). We may take $\nu = 1$, if there is no climate change outside the rod.

Theorem 7.6.9. *The function $v(k_1,k_2)=e_1((k_1)_{\ell_1}).e_1((k_2)_{\ell_2})$ is an exact solution of the x- Fibonacci difference equation (7.41) if the diffusion rate*

$\gamma=\Big[(1-x_1e_1((-\ell_2)_{(\ell_2)})-x_2e_1((-2\ell_2)_{\ell_2}))/(1-x_1e_1(\pm(\ell_1)_{\ell_2})-x_2e_1(\pm(2\ell_1)_{\ell_2}))\Big].$

Proof. By applying the Lemma 7.6.7 on $\underset{x(\pm\ell_1,0)}{\Delta}, \underset{x(\ell_2,0)}{\Delta}$ and (7.8), gives the proof. □

Corollary 7.6.10. *The fractional partial x-Fibonacci difference equation (7.41) of heat flows has a solution for the extorial function $v(k_1,k_2)=e_1((k_1)_{\ell_1}).e_1((k_2)_{\ell_2})$ if*

$\gamma=\Big[(1-x_1e_1((-\ell_2)_{\ell_2})-x_2e_1((-2\ell_2)_{\ell_2}))^{\nu}/(1-x_1e_1(\pm(\ell_1)_{\ell_2})-x_2e_1(\pm(2\ell_1)_{\ell_2}))^{\nu}\Big].$

Assume that $v(k_1,k_2,k_3)$ be the temperature of a thin plate at position (k_1,k_2) at time k_3. Let (ℓ_1,ℓ_2) and ℓ_3 be the shift values of (k_1,k_2) and k_3 respectively and γ be the rate of conductivity of thin plate. A fractional partial x-difference equation of thin plate is

$$\underset{x(\ell_3)}{\Delta^{\nu}}v(k_1,k_2,k_3)=\gamma\left\{\underset{x(\ell_{1,2})}{\Delta^{\nu}}v(k_1,k_2,k_3)+\underset{x(-\ell_{1,2})}{\Delta^{\nu}}v(k_1,k_2,k_3)\right\}. \tag{7.42}$$

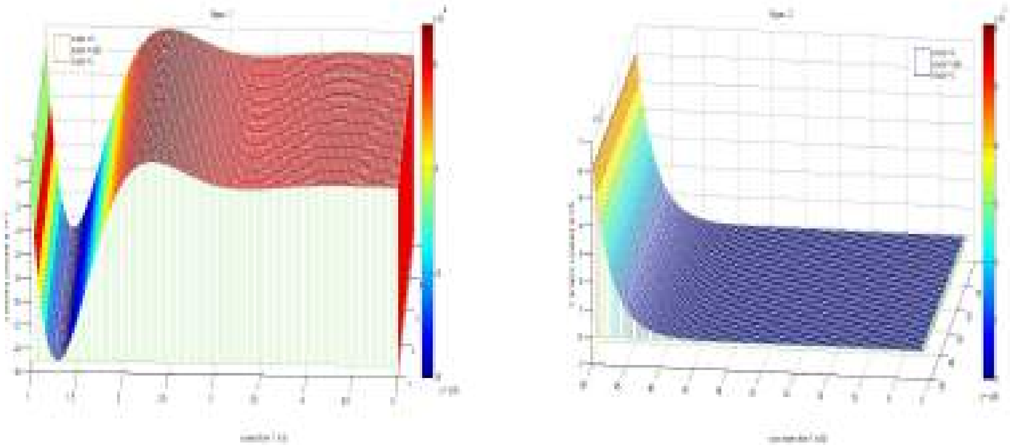
Corollary 7.6.11. $v(k)=\prod_{i=1}^{3}e_1((k_i)_{\ell_i})$ *is a closed form solution of the factional partial x-difference equation (7.42) of heat flow of thin plate for extorial function if*

$\gamma=\Big[(1-x_1e_1((-\ell_3)_{\ell_3})-x_2e_1((-2\ell_3)_{\ell_3}))^{\nu}/(1-x_1e_1(\pm(\ell_{1,2})_{\ell_{1,2}})-x_2e_1(\pm(2\ell_{1,2})_{\ell_{1,2}}))^{\nu}\Big].$

Finally, consider the polynomial factorial function $k_{\ell}^{(n)}$given in (7.2.2) and the difference equation

$$\ell\Delta_{1(\ell)}u(k)\equiv\ell(u(k+\ell)-(1)u(k))=\ell k_{\ell}^{(n-1)}. \tag{7.43}$$

By taking $n = 1$, $\alpha = 1$, $\ell_1 = \ell$, $k_1 = k$ and $u(k) = k_\ell^{(n-1)}$ in (7.37), we get $\underset{1,\ell}{\Delta} k_\ell^{(n)} = (k+\ell)_\ell^{(n)} - k_\ell^{(n)}$, which is same as $k_\ell^{(n-1)}$. Hence, $u(k) = \dfrac{1}{n} k_\ell^{(n)}$ is an exact solution of the equation (7.43). The three dimensional view of the solution of (7.43) is given in figure-1.



From the diagram, we observe that the solution oscillates at initial stage and then it remains constant. Similarly, consider the extorial function $e_1(1,k)$ and for the difference equation

$$\Delta_\ell u(k) \equiv u(k+\ell) - u(k) = \ell e_1(1,k). \tag{7.44}$$

From (ii) of Lemma 7.6.1, $u(k) = e_1(1,k)$ is a solution of equation (7.44) and it is shown in figure-2.

The sum of the solutions of equation (7.43) for $k = 1, 2, , \cdots, 5$ gives solution of equation (7.44). From the diagram, we find that the solution is stable after certain stage. Conversely figure-1 is a decomposition of figure-2, for $k = 1, 2, 3, 4, 5$. Hence, Fourier decomposition can be replaced by this extorial decomposition in filtering the noise in Digital Signal Processing and other related fields.

CONCLUSION

In this book entitled *n-dimensional difference operators in heat equations*, the authors have attempted to develop appropriate difference operators for given situations of heat transfer which would take into consideration the rate of heat transfer and the nature of the material. It also consisted of improvising the already existing operator in order to make it possible to arrive at the solution of fractional order heat equations and an effective formulation of the heat equations with difference operator in order to arrive at the optimal solution.

Here, several difference operators like ℓ operator, q operator, alpha operator, alpha-beta operators, partial operators, and Fibonacci operator have been employed into the study of the heat diffusion. This has helped us get enlightened into the nature of the operators, its working and its limitations. This search led us to tumble on an operator which would have an extended utility and wide comprehensiveness, the extorial function. It is indeed an original and unique contribution to our book.

Secondly, in the area of the heat equations a three-step approach has to be done in order to arrive at the solution of the heat equation of the material taken for study. The first step includes the formation of discrete heat equation using Fourier law of cooling in the field of difference equation. The second step consists in the choice of the appropriate difference operator which would suit our research problem. The third step includes finding the numerical and exact solution of the discrete heat equation and thus analysing it with the use of MATLAB. By the application of the operators mentioned above, the solutions have been derived and the same has been extended to a thin plate and the medium. At the next level, the authors deal with the formulation of delay heat equation model. The partial difference equation which extends its applications in heat equation is taken for study by the application of α-β difference operator and a model for heat transfer in the rod. This newly developed partial alpha-beta difference operator provides relevant results in the field of finite difference methods and the heat equation. The nature of propagation of heat through materials of dimensions up to three are derived using partial difference operator.

The authors proceed further to focuses on solutions of partial difference equation with several variables. Here we have derived some formulae on finite and infinite series of polynomial and rational functions using inverse principle in number theory. We have also derived a new type of discrete q-heat equation model whose solution is a logarithmic function. Heat equation model having sine and cosine functions as solutions are available in the literature. But finding a heat equation model having logarithmic solutions is an interesting and challenging task. Hence, we formulate a discrete q-heat equation model with logarithmic solutions using partial q-difference operator and find the optimal solution. The results of the propagation of heat are also diagrammatically represented. This heat equation model is used to identify the material by proper selection of to avoid heat fluctuation and a decrease in diffusion. The numerical results show that our new technique, described in this paper is an accurate and reliable analytical technique, and it can be extended to high dimensional heat equations with boundary conditions.

The authors proceed on to introduce the partial Fibonacci difference equation and employs it to study the discrete heat equation by having recourse to Fibonacci difference operator with shift values. The operator provides a great possibility to study the various aspects of heat equation: the transfer of heat, nature of the material used and prediction of temperature having the knowledge of the present values as the basis. Here we also introduce a new type of Fibonacci discrete heat equation model for rod using Fibonacci difference operator with delay factor σ. After being successful with the propagation of heat in the rod, the theory has been extended to thin plate and medium after a substantial effort.

After a considerable amount of research dealing with the diffusion of heat in material made up of homogeneous material, an attempt had been made to investigate into the flow of heat in a long rod made of multiple materials stacked together using partial difference equations. The authors present an innovative approach to study the flow of heat in a long rod made of multiple materials stacked together using partial difference equations. The method presented here is very easy accessible for solving the heat equation and determining the temperature for all periods by having the knowledge of initial temperatures. Additionally, the method also provides a useful tool to measure the rate of heat conductivity. The proposed method is efficient and thus can be used to solve the thermal conductivity problems of composite materials.

Finally, the authors make a unique and original contribution by way of introducing a new extorial function using which the solutions of second order difference equation have been obtained in an effective way. Additionally, we have also introduced a new technique for finding the solutions of the fractional difference equation which adds to the significance of this book. On the basis of the above findings, the solution for current flow in the RL circuit is effectively derived after formulating the current flow as the discrete and fractional equations. The uniqueness of this contribution lies in applying the extorial function which makes the process less complicated and more effective.

Hence at the end of this book, we can conclude that we have investigates the generalized partial difference operator and propose a model of it in discrete heat equation with several parameters and shift values. The diffusion of heat is studied in dimensions up to three and several solutions are postulated for the same. At the initial level, the study leads us to the possibility of predicting the temperature either for the past or the future after getting the know the temperature at a few finite points at the present time. It also helps us in making a wise choice of material (γ) for better propagation of heat. In the converse, it also shows the nature of transmission of heat for the material under study. Thus, we can say that the above research helps us in reducing any wastage of heat and also enables us to make an optimal choice of material (γ).

Finally, the authors want to acknowledge that the theory, the results and the applications obtained in this thesis are originally derived. The results incorporated in this thesis have been published in various referred international journals. An innovative attempt has been initiated to make an in-road into the complicated area of heat diffusion in the material using the difference operators, which have been modulated in order to comply to the attempt to find the solutions for heat equations which depict the transfer of heat in the material taken under study. The authors would like to acknowledge that it is only a preliminary attempt and much could be done. A great deal of in-depth study could be undertaken into the numerous aspects that govern the transfer of heat. The proposed method should be upgraded in order to solve the thermal conductivity problems of composite materials. A great deal of unexplored area is available in this area of study and this is only a beginning of a great research journey.

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